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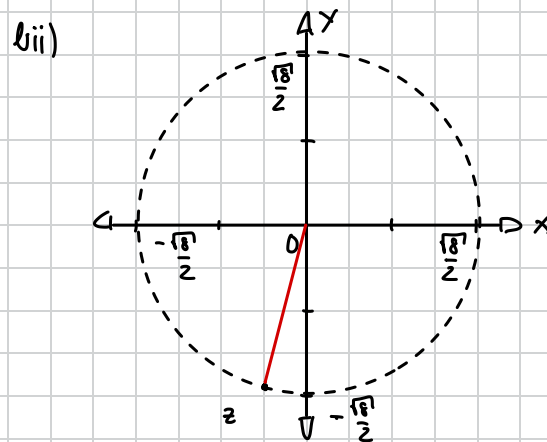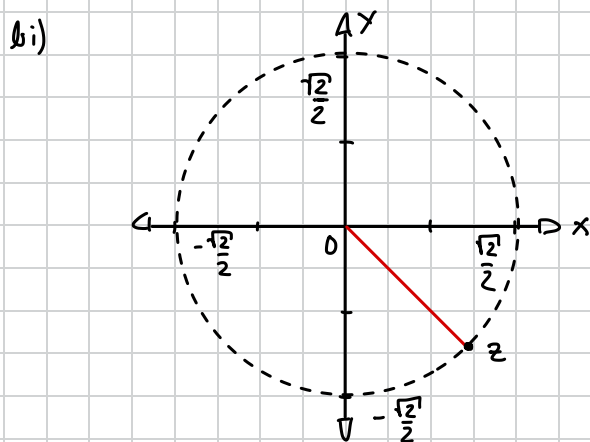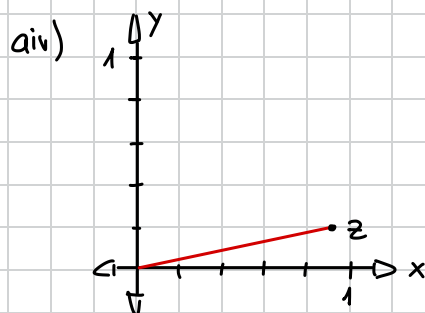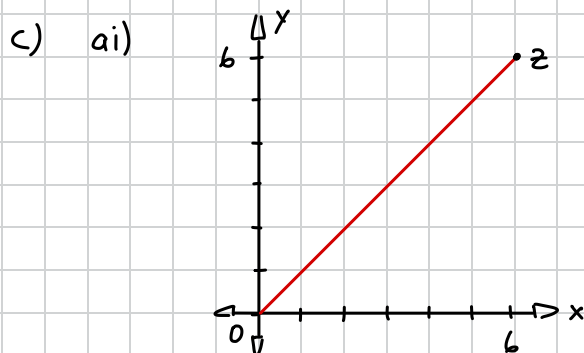<u>Serie 1</u>

Aufgabe 1

a) i) $6 + 6i$ ii) $-10 + i$ iii) $3 - 39i$

iv) $\dfrac{(12 + 3i)}{(12 + 6i)} = \dfrac{(12 + 3i)(12 - 6i)}{144 + 36} = \dfrac{162 - 36i}{180}$
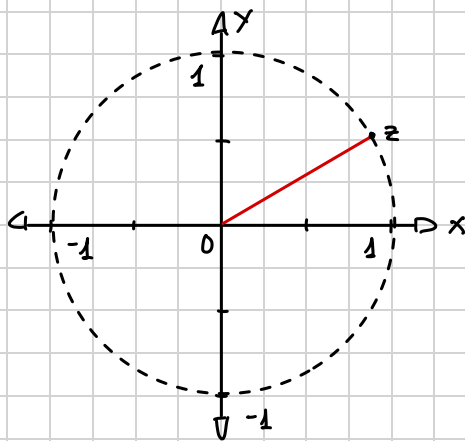
b) i) $-i \;\to\; 1\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

   $1 - i \;\Rightarrow\; \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

   $|z| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

   $\arg(z) = \left(-\frac{\pi}{2} + \frac{\pi}{4}\right) = -\frac{\pi}{4}$

   $\Rightarrow z = \dfrac{\sqrt{2}}{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

ii) $-2 - 2i \;\Rightarrow\; \sqrt{8}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

   $1 + \sqrt{3}i \;\Rightarrow\; 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

   $|z| = \dfrac{\sqrt{8}}{2}$

   $\arg(z) = \left(-\frac{3\pi}{4} - \frac{\pi}{3}\right) = -\frac{13\pi}{12} = \frac{11\pi}{12}$

   $\Rightarrow z = \dfrac{\sqrt{8}}{2}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right)$

iii) $1 + \sqrt{3}i \;\Rightarrow\; 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

   $\sqrt{3} + i \;\Rightarrow\; 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

   $|z| = 1$

   $\arg(z) = \left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{6}$

   $z = 1\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

c) ai)

aiv)

bi)
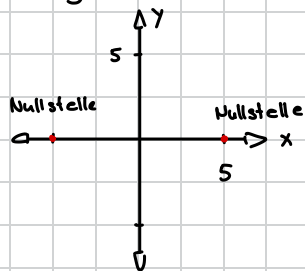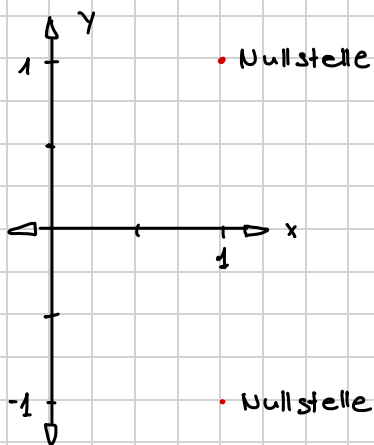

bii)

b)iii)

Aufgabe 2

i) $z = \pm 5$

ii) $z^2 - 2z + 2 \overset{!}{=} 0$

$z_{1,2} = 1 \pm \sqrt{1 - 2}$

$= 1 \pm i$

iii) $z^3 + z^2 - 2 : z - 1 = z^2 + 2z + 2$

$\underline{z^3 - z^2}$

$\quad 2z^2$

$\quad \underline{2z^2 - 2z}$

$\qquad 2z - 2$

$\qquad \underline{2z - 2}$

$z_1 = 1$

$z^2 + 2z + 2 \overset{!}{=} 0$

$z_{2,3} = -1 \pm \sqrt{1 - 2}$

$= -1 \pm i$

iv) $z = 1 (\cos(7\varphi) + i \cdot \sin(7\varphi))$
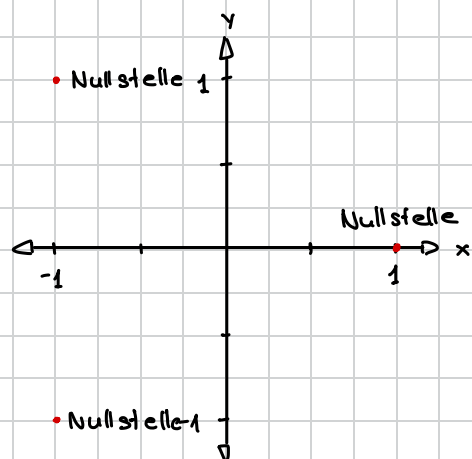
$7\varphi = 2\pi n \Rightarrow \varphi = \frac{2\pi n}{7}$

Aufgabe 1

i) $M = \{z \in \mathbb{C} \mid |z| = 3, \operatorname{Im}(z) \geqslant 0\}$

ii) $M = \left\{ z \in \mathbb{C} \mid \dfrac{|z+2-2i|}{|z+i|} = 2 \right\}$

$= \left\{ z \in \mathbb{C} \mid \dfrac{|x+2+i(y-2)|}{|x+i(y+1)|} = 2 \right\}$
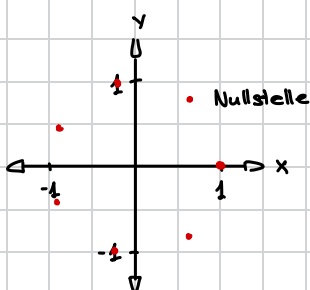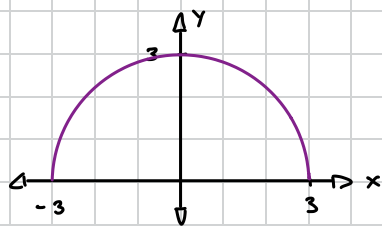
$\left(|x+2+i(y-2)|\right)^2 = \left(2\,|x+i(y+1)|\right)^2$

$(x+2)^2 + (y-2)^2 = 4(x^2 + (y+1)^2)$

$x^2 + 4x + 4 + y^2 - 4y + 4 = 4x^2 + 4y^2 + 8y + 4$

$-3x^2 + 4x + 4 = 3y^2 + 12y$

$3x^2 - 4x + 3y^2 + 12y = 4$

$x^2 - \tfrac{4}{3}x + y^2 + 4y = \tfrac{4}{3}$
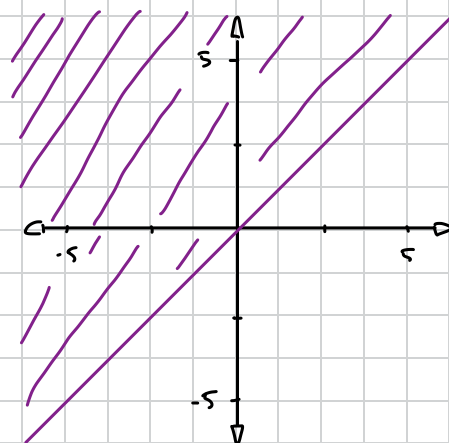
$\left(x - \tfrac{2}{3}\right)^2 + (y+2)^2 = \tfrac{52}{9}$

$M = \left\{ z \in \mathbb{C} \mid \left| z - \tfrac{2}{3} + 2i \right| = \dfrac{2\sqrt{13}}{3} \right\}$
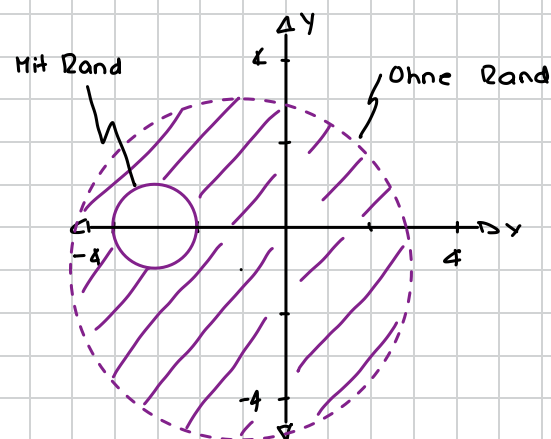
Nur der Rand
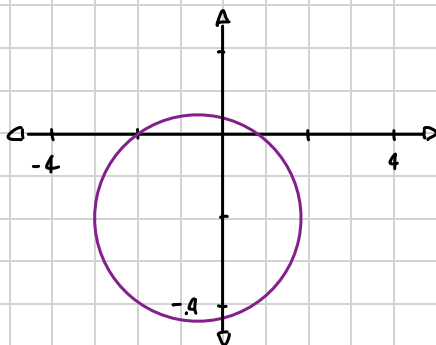
iii) $M = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geqslant \operatorname{Re}(z)\}$

iv) $M = \{z \in \mathbb{C} \mid |z-3| \geqslant 1 \text{ und } |z-1-i| < 4\}$

Mit Rand

Ohne Rand

Serie 3

Aufgabe 1

a)

i) $\lim\limits_{|n|\to\infty} \cos(in) = \lim\limits_{|n|\to\infty} \cosh(n) = \underline{\underline{\infty}}$

ii) $\lim\limits_{n\to\infty} 1+(-1)^n \cdot \frac{i}{n} = 1 + \lim\limits_{n\to\infty} \left(-1^n \cdot \frac{i}{n}\right) = 1+0i = \underline{\underline{1}}$ $\|$ Wenn $n\to\infty$ wird $\frac{i}{n}\to 0$

iii) $\lim\limits_{n\to\infty} \frac{(n+2\pi i)^n}{n^n} = \lim\limits_{n\to\infty} \frac{n^n}{n^n} = \underline{\underline{1}}$ $\Big\|$ $2\pi i$ hat kein Einfluss aufs Ergebnis wenn $n\to\infty$

iv) $\lim\limits_{n\to\infty} \text{Arg}\left(1+(-1)^n \frac{i}{n}\right)$ $\Big\|$ $\lim\limits_{n\to\infty} 1+(-1)^n \frac{i}{n} = 1$

$\text{Arg}(1) = \underline{\underline{0}}$

b)

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n} + \frac{(\pi i)^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{1}{2^n} + \sum_{n=0}^{\infty}\frac{(\pi i)^n}{n!} = 2 + \exp(\pi i) = \underline{\underline{2 + e^{\pi i}}}$$

Aufgabe 2

$\exp(it) = \cos(t) + i\cdot\sin(t)$

$\exp(z_1 + z_2) = \exp(z_1)\cdot\exp(z_2)$ $\|$ $z_1 = ix$; $z_2 = iy$

$\exp(i(x+y)) = \exp(ix)\cdot\exp(iy)$ $\|$ Eulerformel

$\exp(i(x+y)) = (\cos(x)+i\sin(x))\cdot(\cos(y)+i\sin(y))$

$\cos(x+y) + i\sin(x+y) = (\cos(x)+i\sin(x))\cdot(\cos(y)+i\sin(y))$

$\cos(x+y) + i\sin(x+y) = \cos(x)\cos(y) + \cos(x)i\sin(y) + i\sin(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x+y) + i\sin(x+y) = (\cos(x)\cos(y) - \sin(x)\sin(y)) + i(\cos(x)\sin(y) + \cos(y)\sin(x))$

$\Rightarrow \cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ $\Big\|$ Ich bin zu faul um es für $x-iy$ zu beweisen :-)

$\Rightarrow \sin(x+y) = \cos(x)\sin(y) + \cos(y)\sin(x)$

$\square$

Aufgabe 3

i) $\lim\limits_{z\to 0} \frac{\bar{z} + z^2}{z}$ $\Big\|$ $z = x+iy$

$\lim\limits_{z\to 0} \frac{x-iy + x^2 + 2xiy - y}{x+iy}$ $\Big\|$ Wir müssen den Grenzwert von Re und Im seperat betrachten

$\lim\limits_{y\to 0} \frac{x - iy + x^2 + 2xiy - y^2}{x+iy}$ $\Big\|$ $x = 0$

$\lim\limits_{y\to 0} \frac{-iy - y^2}{iy} = \lim\limits_{y\to 0} -1 - \frac{y^2}{iy} = \lim\limits_{y\to 0} -1 - \frac{y}{i} = \underline{\underline{-1}}$

Aufgabe 3

$$\lim_{x \to 0} \frac{x - iy + x^2 + 2xiy - y^2}{x + iy} \quad \Big\| \; y = 0$$

$$\lim_{x \to 0} \frac{x + x^2}{x} = \lim_{x \to 0} 1 + x = 1$$

$$z_0 = 1 - i \neq 0 \quad \| \text{ Somit existiert der Limes nicht.}$$

ii)

$$\lim_{z \to 0} \frac{\cos(z) - 1}{z^2} \quad \Big\| \; \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \quad \text{(Taylorreihe)}$$

$$\lim_{z \to 0} \frac{-\frac{z^2}{2!} + \frac{z^4}{4!} + \dots}{z^2} = \underline{\underline{-\frac{1}{2}}}$$

iii)

$$\lim_{z \to 0} \frac{\sin(z)}{\bar{z}} \quad \Big\| \; z = x + iy$$

$$\lim_{z \to 0} \frac{\sin(x + iy)}{x - iy} \quad \Big\| \; \text{Wir müssen den Grenzwert von Re und Im seperat betrachten}$$

$$\lim_{x \to 0} \frac{\sin(x + iy)}{x - iy} \quad \Big\| \; y = 0$$

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad \Big\| \; \lim_{x \to 0} \frac{\sin(x)'}{x'} = \frac{\cos(x)}{1} = 1 \quad \text{(Bernoulli - l'Hôpital)}$$

$$\lim_{y \to 0} \frac{\sin(x + iy)}{x - iy} \quad \Big\| \; x = 0$$

$$\lim_{y \to 0} \frac{\sin(iy)}{-iy} = \lim_{y \to 0} \frac{i \cdot \sin(y)}{-iy} = \lim_{y \to 0} -\frac{\sin(y)}{y} = -1 \quad \Big\| \; \text{Bernoulli - l'Hôpital}$$

$$\Rightarrow z_0 = 1 - i \neq 0 \quad \| \text{ Somit existiert der Limes nicht.}$$

<u>Serie 4</u>

Aufgabe 1

a) i) $\underline{e^i = \cos(1) + i\sin(1)}$

ii) $e^{1-2i} = e \cdot e^{-2i}$

$e^{-2i} = \cos(-2) + i\sin(-2)$

$\Rightarrow \underline{\underline{e^{1-2i} = e\cos(-2) + e \cdot i\sin(-2)}}$

iii) $\text{Log}(1+i) = \log(\sqrt{2}) + i\frac{\pi}{4}$

b) i) $\cos(10i) = \cosh(10)$

$\cosh(10) = \dfrac{e^x + e^{-x}}{2}$

ii) $\sin(5+5i) = \sin(5)\cos(5i) + \cos(5)\sin(5i)$

$\cos(5i) = \cosh(5) = \dfrac{e^5 + e^{-5}}{2}$

$\sin(5i) = \sinh(5) = \dfrac{e^5 - e^{-5}}{2}$

$\underline{\underline{\sin(5+5i) = \sin(5)\dfrac{e^5+e^{-5}}{2} + \cos(5)\dfrac{e^5-e^{-5}}{2}}}$

iii) $\sin(2-i) = \sin(2)\cos(i) - \cos(2)\sin(i)$

$\cos(i) = \dfrac{e^1 + e^{-1}}{2}$

$\sin(i) = \dfrac{e^1 - e^{-1}}{2}$

$\underline{\underline{\sin(2-i) = \sin(2)\dfrac{e^1+e^{-1}}{2} - \cos(2)\dfrac{e^1-e^i}{2}}}$

Aufgabe 2

$\text{Log}(z_1 \cdot z_2) \neq \text{Log}(z_1) + \text{Log}(z_2)$

$\Rightarrow \begin{cases} z_1 = -1 \\ z_2 = i \end{cases}$

‖ Wir suchen z_1 und z_2, sodass die Summe der Arg() $> \pi$ sind aber das Arg() von $z_1 \cdot z_2$ kleiner als π ist.

Aufgabe 3

$(\cos(\phi) + i\sin(\phi))^n = \cos(n\phi) + i \cdot \sin(n\phi)$ || Beweis durch Induktion

n = 0

$(\cos(\phi) + i\sin(\phi))^n = \cos(n\phi) + i \cdot \sin(n\phi)$

$1 = 1 + 0$ ✓

n = 1

$(\cos(\phi) + i\sin(\phi))^1 = \cos(\phi) + i\sin(\phi)$ ✓

n = k + 1

$(\cos(\phi) + i\sin(\phi))^{k+1} = (\cos(\phi) + i\sin(\phi))^k \cdot (\cos(\phi) + i\sin(\phi))$

$= (\cos(k\phi) + i\sin(k\phi))(\cos(\phi) + i\sin(\phi))$

$= \cos(k\phi)\cos(\phi) + \cos(k\phi)i\sin(\phi) + i\sin(k\phi)\cos(\phi) - \sin(k\phi)\sin(\phi)$ || Kosinus & Sinus Satz

$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$\sin(A+B) = \cos(A)\sin(B) + \cos(B)\sin(A)$

$\Rightarrow \cos(k\phi + k) + \sin(k\phi + k) = \cos((k+1)\phi) + \sin((k+1)\phi)$

▢

Aufgabe 1

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad \| \quad \Delta z = z - z_0$$

i) $f(z) = 3z^3 + z - 3$

$$f'(z) = \lim_{\Delta z \to 0} \frac{3(z + \Delta z)^3 + (z + \Delta z) - 3 - (3z^3 + z - 3)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3(z^3 + 3z^2 \Delta z + 3z \Delta z^2 + \Delta z^3) + (z + \Delta z) - 3 - (3z^3 + z - 3)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3z^3 + 9z^2 \Delta z + 9z \Delta z^2 + 3\Delta z^3 + z + \Delta z - 3 - 3z^3 - z + 3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{9z^2 \Delta z + 9z \Delta z^2 + 3\Delta z^3 + \Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z (9z^2 + 9z \Delta z + 3\Delta z^2 + 1)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 9z^2 + 9z \Delta z + 3\Delta z^2 + 1$$

$$\Rightarrow \underline{\underline{f'(z) = 9z^2 + 1}}$$

ii) $f(z) = \sin(\operatorname{Re}(z)) = \sin(x)$ $\|$ Grundsätzlich wäre diese Gleichung ableitbar aber es ist in einem Bereich ableitbar.

$$f'(z) = \cos(x)$$

$$\tilde{x} \in \left[\frac{\pi}{2} + k\pi \right]$$

$\Rightarrow$ Wenn $x \in \tilde{x}$ dann existiert die Ableitung, sonst nicht

iii) $f(z) = \dfrac{1}{z^2}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\frac{1}{(z + \Delta z)^2} - \frac{1}{z^2}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\frac{z^2 - (z + \Delta z)^2}{z^2(z + \Delta z)^2}}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 - (z + \Delta z)^2}{\Delta z (z^2(z + \Delta z)^2)}$$

$$= \lim_{\Delta z \to 0} \frac{z^2 - z^2 - 2z\Delta z - \Delta z^2}{\Delta z (z^2(z + \Delta z)^2)} = \lim_{\Delta z \to 0} \frac{\Delta z (-2z - \Delta z)}{\Delta z (z^2(z + \Delta z)^2)}$$

$$\Rightarrow \underline{\underline{f'(z) = \frac{-2z}{z^4} = -\frac{2}{z^3}}}$$

iv) $f(z) = e^{-\pi z^2}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{e^{-\pi(z+\Delta z)^2} - e^{-\pi z^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{e^{-\pi(z^2 + 2z\Delta z + \Delta z^2)} - e^{-\pi z^2}}{\Delta z} = \lim_{\Delta z \to 0} \frac{e^{-\pi z^2} \cdot e^{-\pi(2z\Delta z + \Delta z^2)} - e^{-\pi z^2}}{\Delta z}$$

$\Rightarrow$ Wir verwenden die Aproximation, dass $e^{-\pi(2z\Delta z + \Delta z^2)} \approx 1 - \pi(2z\Delta z + \Delta z^2)$ für

$\Delta z \to 0$ (Taylor Series $\to e^x = 1 + x + \frac{x^2}{2!} + \dots$)

$$= \lim_{\Delta z \to 0} \frac{e^{-\pi z^2}(1 - \pi(2z\Delta z + \Delta z^2) - 1)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{e^{-\pi z^2}(-\pi(2z\Delta z + \Delta z^2))}{\Delta z}$$

$$\Rightarrow \underline{\underline{f'(z) = -2\pi z \cdot e^{-\pi z^2}}}$$

Aufgabe 2

$g(t) = f(\gamma(t))$

$$g'(t) = \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(\gamma(t + \Delta t)) - f(\gamma(t))}{\Delta t} \quad \Bigg\| \cdot \frac{\gamma(t+\Delta t) - \gamma(t)}{\gamma(t+\Delta t) - \gamma(t)}$$

$$= \lim_{\gamma(t+\Delta t) \to \gamma(t)} \frac{f(\gamma(t+\Delta t)) - f(\gamma(t))}{\gamma(t+\Delta t) - \gamma(t)} \cdot \lim_{\Delta t \to 0} \frac{\gamma(t+\Delta t) - \gamma(t)}{\Delta t} \quad \Bigg\| \Delta\gamma = \gamma(t+\Delta t) - \gamma(t) \Rightarrow \Delta z = z - z_0$$

$$\Rightarrow \underline{\underline{g'(t) = f'(\gamma(t)) \cdot \dot\gamma(t)}}$$

Aufgabe 3

Cauchy-Riemann Gleichung: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$; $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

a) $u(x,y) := \sin(x^2 - y^2)\cosh(2xy)$

$$\frac{\partial u}{\partial x} = 2x \cdot \cos(x^2 - y^2)\cosh(2xy) + 2y \cdot \sin(x^2 - y^2)\sinh(2xy)$$

$$\frac{\partial u}{\partial y} = -2y \cdot \cos(x^2 - y^2)\cosh(2xy) + 2x \cdot \cos(x^2 - y^2)\sinh(2xy)$$

$v(x,y) = -\cos(x^2 - y^2)\sinh(2xy)$

$$\frac{\partial v}{\partial x} = 2x \cdot \sin(x^2 - y^2)\sinh(2xy) - 2y\cos(x^2 - y^2)\cosh(2xy)$$

$$\frac{\partial v}{\partial y} = -2y \cdot \sin(x^2 - y^2)\sinh(2xy) - 2x \cdot \cos(x^2 - y^2)\cosh(2xy)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \checkmark$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \checkmark$$

$\Rightarrow$ Da die Cauchy-Riemann Gleichungen gelten, ist die Gleichung holomorph.

b) $u(x,y) := e^{x^2-y^2} \cos(2xy)$

$$\frac{\partial u}{\partial x} = 2x \cdot e^{x^2-y^2} \cdot \cos(2xy) - 2y \cdot e^{x^2-y^2} \cdot \sin(2xy)$$

$$\frac{\partial u}{\partial y} = -2y \cdot e^{x^2-y^2} \cdot \cos(2xy) - 2x \cdot e^{x^2-y^2} \cdot \sin(2xy)$$

$v(x,y) := e^{x^2-y^2} \sin(2xy)$

$$\frac{\partial v}{\partial x} = 2x \, e^{x^2-y^2} \cdot \sin(2xy) + 2y \cdot e^{x^2-y^2} \cdot \cos(2xy)$$

$$\frac{\partial v}{\partial y} = -2y \cdot e^{x^2-y^2} \cdot \sin(2xy) + 2x \cdot e^{x^2-y^2} \cdot \cos(2xy)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \checkmark$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \checkmark$$

$\Rightarrow$ Da die Cauchy-Riemann Gleichung gelten, ist die Gleichung holomorph

Aufgabe 4

$x = r \cdot \sin(\varphi)$

$y = r \cdot \cos(\varphi)$

$r = \sqrt{x^2+y^2}$

$\varphi = \arctan\left(\frac{x}{y}\right)$

$\tilde{u}(x,y) = \tilde{u}(r \cdot \sin(\varphi), r \cdot \cos(\varphi))$

$\tilde{v}(x,y) = \tilde{v}(r \cdot \sin(\varphi), r \cdot \cos(\varphi))$

$$\frac{\partial \tilde{u}}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial r} \quad \| \quad \text{Kettenregel}$$

$$\frac{\partial \tilde{u}}{\partial \varphi} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial \varphi}$$

$$\frac{\partial \tilde{v}}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial v}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial r}$$

$$\frac{\partial \tilde{v}}{\partial \varphi} = \frac{\partial v}{\partial x}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial \varphi}$$

$$\frac{\partial \tilde{u}}{\partial r} = \frac{\partial u}{\partial x}\cdot \cos(\varphi) + \frac{\partial u}{\partial y}\cdot \sin(\varphi) \quad \Longrightarrow \quad \frac{\partial v}{\partial y}\cdot \cos(\varphi) - \frac{\partial v}{\partial x}\cdot \sin(\varphi)$$

$$\frac{\partial \tilde{u}}{\partial \varphi} = \frac{\partial u}{\partial x}\cdot -r\cdot \sin(\varphi) + \frac{\partial u}{\partial y}\cdot r\cdot \cos(\varphi) \quad \Longrightarrow \quad \frac{\partial v}{\partial y}\cdot -r\cdot \sin(\varphi) - \frac{\partial v}{\partial x}\cdot r\cdot \cos(\varphi)$$

$$\frac{\partial \tilde{v}}{\partial r} = \frac{\partial v}{\partial x}\cdot \cos(\varphi) + \frac{\partial v}{\partial y}\cdot \sin(\varphi) \quad \Longrightarrow \quad -\frac{\partial u}{\partial y}\cdot \cos(\varphi) + \frac{\partial u}{\partial x}\cdot \sin(\varphi)$$

$$\frac{\partial \tilde{v}}{\partial \varphi} = \frac{\partial v}{\partial x}\cdot -r\cdot \sin(\varphi) + \frac{\partial v}{\partial y}\cdot r\cdot \cos(\varphi) \quad \Longrightarrow \quad -\frac{\partial u}{\partial y}\cdot -r\cdot \sin(\varphi) + \frac{\partial u}{\partial x}\cdot r\cdot \cos(\varphi)$$

$$r\cdot \frac{\partial \tilde{u}}{\partial r} = r\left(\frac{\partial v}{\partial y}\cdot \cos(\varphi) - \frac{\partial v}{\partial x}\cdot \sin(\varphi)\right) = \frac{\partial v}{\partial y}\cdot r\cdot \cos(\varphi) - \frac{\partial v}{\partial x}\cdot r\cdot \sin(\varphi) = \frac{\partial \tilde{v}}{\partial r} \quad \checkmark$$

$$-r\cdot \frac{\partial \tilde{v}}{\partial r} = -r\left(-\frac{\partial u}{\partial y}\cdot \cos(\varphi) + \frac{\partial u}{\partial x}\cdot \sin(\varphi)\right) = \frac{\partial u}{\partial y}\cdot r\cdot \cos(\varphi) - \frac{\partial u}{\partial x}\cdot \sin(\varphi) = \frac{\partial \tilde{u}}{\partial \varphi} \quad \checkmark$$

$\square$

Aufgabe 1

i) $f(t) = z_0 + t(z_1 - z_0)$

$= i + t(4+4i)$

$\Rightarrow f(t) = 4t + i(1 + 4t)$

ii) $f(t) = z_0 + r \cdot e^{it}$

$= 3 - 2i + 5(\cos(t) + i \cdot \sin(t))$

$\Rightarrow f(t) = 3 + 5\cos(t) + i(\sin(t) - 2)$

Aufgabe 2

i) $\int_\gamma f(z) + g(z)\, dz = \int_0^1 (f(\gamma(t)) + g(\gamma(t))) \cdot \gamma'(t)\, dt$ $\quad \Bigg\|$ Summenregel für Integrale: $\int f(x) + g(x)\, dx = \int f(x)dx + \int g(x)dx$

$= \int_0^1 f(\gamma(t)) \cdot \gamma'(t)\, dt + \int_0^1 g(\gamma(t)) \cdot \gamma'(t)\, dt = \int_\gamma f(z)\, dz + \int_\gamma g(z)\, dz$ $\quad \square$

ii) $|\gamma| := \int_0^1 |\dot\gamma(t)|\, dt$

$\left| \int_\gamma f(z)\, dz \right| = \left| \int_0^1 f(\gamma(t)) \cdot \gamma'(t)\, dt \right| \leq \int_0^1 |f(\gamma(t)) \cdot \gamma'(t)|\, dt = \int_0^1 |f(\gamma(t))| \cdot |\gamma'(t)|\, dt$

Sei $|f(\gamma(t))|$ maximal $(\max_{t\in[0,1]} |f(\gamma(t))|$

$\Rightarrow \max_{t\in[0,1]} |f(\gamma(t))| \cdot \int_0^1 \gamma'(t)\, dt = \max_{t\in[0,1]} \cdot \gamma$

Und es gilt auch $\max_{t\in[0,1]} |f(\gamma(t))| = \max_{z\in\gamma([0,1])} |f(z)|$

(Beide Aussagen sind äquivalent) $\quad \square$

iii) $\overline{\int_\gamma f(z)\, dz} = \overline{\int_0^1 f(e^{2\pi it}) \cdot 2\pi i e^{2\pi it}\, dt} = \int_0^1 \overline{f(e^{2\pi it})} \cdot \overline{2\pi i e^{2\pi it}}\, dt$

$= \int_0^1 \overline{f(e^{2\pi it})} \cdot (-2\pi i e^{-2\pi it})\, dt$ $\quad \Bigg\| \overline{2\pi i} = -2\pi i$ und $\overline{e^{2\pi it}} = \overline{\cos(2\pi t) + i\sin(2\pi t)}$ $= \cos(2\pi t) - i\sin(2\pi t) = \cos(-2\pi t) + i\sin(-2\pi t)$

$= -\int_0^1 \overline{f(z)} \cdot (2\pi i e^{-2\pi it})\, dt$ $\quad \Bigg\| e^{-2\pi it} = \dfrac{e^{2\pi it}}{e^{4\pi it}}$

$= -\int_0^1 \dfrac{\overline{f(z)}}{z^2} \cdot 2\pi i e^{2\pi it}\, dt = \int_\gamma \dfrac{\overline{f(z)}}{z^2}\, dz$ $\quad \square$

Aufgabe 3

$$\int_\gamma \mathrm{Im}(z)\, dz = \int_0^{2\pi} \mathrm{Im}(z)\, z'(t)\,dt = \int_0^{2\pi} \sin(t)\cdot(-\sin(t)+i\cos(t))\,dt = \int_0^{2\pi} -\sin(t)^2 + i\sin(t)\cos(t)\,dt$$

$$\underset{\shortparallel}{}$$

$$\frac{dz}{dt} = z'(t)$$

$$\Rightarrow \int_0^{2\pi} -\sin(t)^2\, dt = -\int_0^{2\pi} \frac{1-\cos(t)}{2}\,dt = -\left[\frac{t}{2} - \frac{\sin(2t)}{4}\right]_0^{2\pi} = -\pi$$

$$\Rightarrow \int_0^{2\pi} i\cdot\sin(t)\cdot\cos(t)\,dt = i\int_0^{2\pi} \frac{1}{2}\cdot\sin(2t)\,dt = \frac{i}{2}\left[-\frac{\cos(2t)}{2}\right]_0^{2\pi} = 0$$

$$\Rightarrow -\pi + 0 = \underline{\underline{-\pi}}$$

Aufgabe 4

$$\int_\gamma \frac{1}{z^n}\,dz \qquad \Big\| \quad z(t) = \cos(t) - i\cdot\sin(t) = e^{-it}\ (\text{Uhrzeigersinn})$$

$$= \int_0^{2\pi} \frac{1}{z^n}\cdot z'(t)\,dt = \int_0^{2\pi} e^{int}\cdot -i\cdot e^{-it}\,dt = \int_0^{2\pi} -i\cdot e^{int-it}\,dt = \int_0^{2\pi} -i\cdot e^{(n-1)it}\,dt$$

$$\Rightarrow n = 1$$

$$\rightarrow \int_0^{2\pi} -i\,dt = [-it]_0^{2\pi} = -i2\pi$$

$$\Rightarrow n \neq 1$$

$$\rightarrow \int_0^{2\pi} -i\cdot e^{(n-1)it} = -i\left[\frac{e^{(n-1)it}}{(n-1)i}\right]_0^{2\pi} = 1-1 = 0 \qquad \Big\| \quad e^{i2\pi(n-1)} = \cos((n-1)2\pi) + i\sin((n-1)2\pi)$$

$$\Rightarrow -i2\pi - 0 = \underline{\underline{-i2\pi}}$$

Serie 7

Aufgabe 1

$$\int_\gamma \cos\left(\frac{z}{2}\right) dz = 0$$

|| Das Wegintegral ist null, da ein geschlossener Weg ist.

Aufgabe 2

a)
$$\int z^n \, dz = \frac{z^{(n+1)}}{n+1} + c$$

Für $n = -1$

$$z^n = \frac{1}{z}$$

$$\int \frac{1}{z} \, dz = Log(z)$$ || $Log(z) \in (-\infty, 0]$ welches nicht im Bereich $\mathbb{C} \setminus \{0\}$

$$\Rightarrow \begin{cases} \dfrac{z^{(n+1)}}{n+1} & \text{für } n \neq -1 \\ \text{existiert nicht} & \text{für } n = -1 \end{cases}$$

b) Siehe a) für Rechenweg

$$\Rightarrow \begin{cases} \dfrac{z^{(n+1)}}{n+1} & \text{für } n \neq -1 \\ Log(z) & \text{für } n = -1 \end{cases}$$

c) $\exp(l(z)) = z$

$\exp(l(z)) \cdot l'(z) = 1$ || $\exp(l(z)) = z$

$l'(z) = \frac{1}{z}$ || von a) wissen wir, dass

□

Aufgabe 3

a) Von CRDG wissen wir, dass $\dfrac{\partial u}{\partial y}(x,y) = -\dfrac{\partial v}{\partial x}(x,y)$

$\Rightarrow z = x + iy$

$f(z) = u(x,y) + iv(x,y)$

$f'(z) = \dfrac{\partial u}{\partial x}(x,y) + i\dfrac{\partial v}{\partial x}(x,y)$ || CRDG

$\underline{\underline{f'(z) = \dfrac{\partial u}{\partial x}(x,y) - i\dfrac{\partial u}{\partial y}(x,y)}}$

b)

Wir nehmen an, dass $g(z) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y) = P(x,y) + Q(x,y)$

$$\frac{\partial P}{\partial x}(x,y) = \frac{\partial Q}{\partial y}(x,y)$$

$$\frac{\partial P}{\partial y}(x,y) = -\frac{\partial Q}{\partial x}(x,y)$$

$\Big\}$ CRDG

$$\frac{\partial P}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial Q}{\partial y} = -\frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}$$ || Gegeben ist $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} \quad \checkmark$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial Q}{\partial x} = -\frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$ || Wir nehmen an, dass die Funktion stetig ist

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial x \partial y} \quad \checkmark$$

c) f ist holomorph $\rightarrow$ Es existiert eine Ableitung g, so dass $f(z) \int g(z)\,dz$

$$f'(z) = g(z)$$

$$\frac{\partial u_1}{\partial x} - i\frac{\partial u_1}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} \rightarrow \int \frac{\partial u_1}{\partial x} = \int \frac{\partial u}{\partial x} \rightarrow u_1(x,y) = u(x,y) + c$$

Wenn $c = 0$ ist, so gilt $u_1(x,y) = u(x,y)$

$$\rightarrow Re(f(z)) = u(x,y) \rightarrow Re(f) = u$$

d) Nein, da $i\int \frac{\partial v}{\partial y} = iv(x,y) + iC$ wobei C beliebig gewählt werden kann.

DISCLAIMER

Die Notizen zu den Stack Aufgaben sind auf meine Werte angepasst. Die Werte können von deinen Aufgaben abweichen.

Frage 1

$(8 - i)(5i + 3) = 40i + 24 + 5 - 3i = 37i + 29$

$i(5i + 3) = -5 + 3i$

$\bar{w} = -5i + 3$

$\dfrac{8 - i}{5i + 3} = \dfrac{(8 - i)(5i - 3)}{-34} = \dfrac{40i - 24 + 5 + 3i}{-34} = -\dfrac{43i}{34} + \dfrac{19}{34}$

$|z| \quad \sqrt{8^2 + 1^2} = \sqrt{65}$

Frage 2

$\cos\left(-\dfrac{\pi}{4}\right) > 0 \; ; \; \cos\left(\dfrac{\pi}{4}\right) > 0 \quad \Rightarrow \quad Re(z) > 0$

$\sin\left(-\dfrac{\pi}{4}\right) < 0 \; ; \; \sin\left(\dfrac{\pi}{4}\right) > 0 \quad \Rightarrow \quad Im(z) > / = / < 0$

Frage 3

$(4 + 2i)(4 + 2i) = 16 + 8i + 8i - 4 = 12 + 16i$

$(12 + 16i)(4 + 2i) = 48 + 24i + 64i - 32 = 16 + 88i$

Bemerkung: Teilweise ist es einfacher potenzierte komplexe Gleichungen

in Polarform zu berechnen.

Serie 2

Serie 3

Frage 1

$\text{Im}(z) = 0$ ∥ Der Im von z muss 0 sein damit $z \in \mathbb{R}$

$z = e^{\frac{3\pi}{4}i} \cdot (\sqrt{2} + bi)$

$= \left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) \cdot (\sqrt{2} + bi)$

$= \sqrt{2}\cos\left(\frac{3\pi}{4}\right) + bi\cos\left(\frac{3\pi}{4}\right) + \sqrt{2}\,i\sin\left(\frac{3\pi}{4}\right) + b\sin\left(\frac{3\pi}{4}\right)$ ∥ $\text{Im}(z) = 0$

$\Rightarrow bi\cos\left(\frac{3\pi}{4}\right) + \sqrt{2}\,i\sin\left(\frac{3\pi}{4}\right) = 0$

$bi\cos\left(\frac{3\pi}{4}\right) = -\sqrt{2}\,i\sin\left(\frac{3\pi}{4}\right)$

$-bi\frac{\sqrt{2}}{2} = -i$

$b = \frac{2\sqrt{2}}{2}$

Frage 2

$c_n = \dfrac{(4n)!}{(-2-i)^n \cdot (n!)^4}$

$R = \dfrac{1}{\limsup\limits_{n \to \infty} \sqrt[n]{|c_n|}}$

$\limsup\limits_{n \to \infty} \sqrt[n]{\left|\dfrac{(4n)!}{(-2-i)^n \cdot (n!)^4}\right|} = \limsup\limits_{n \to \infty} \sqrt[n]{\left|\dfrac{1}{\sqrt{5}^n}\right| \dfrac{(4n)!}{(n!)^4}}$

$(4n)! = (4n)(4n-1)(4n-2)\ldots 4 \cdot 3 \cdot 2 \cdot 1$

Wenn wir die Multiplikation in 4er Gruppen aufteilen:

$\left.\begin{array}{l} [(4n)(4n-1)(4n-2)(4n-3)] \leq (4n)^4 \\ \vdots \\ [4 \cdot 3 \cdot 2 \cdot 1] \leq (4n)^4 \end{array}\right\} n\text{-Mal}$

$\Rightarrow (4n)! \leq (4n)^4 \cdot (4n)^4 \cdots = (4n)^{4n}$

$(n!)^4 = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

Wenn wir die Multiplikation in 4er Gruppen aufteilen

$\left.\begin{array}{l} [n \cdot (n-1)(n-2)(n-3)] \leq n^4 \\ \vdots \\ [4 \cdot 3 \cdot 2 \cdot 1] \leq n^4 \end{array}\right\} n\text{-Mal}$

$\Rightarrow (n!)^4 \leq n^{4n}$

Frage 2

$$\limsup_{n\to\infty} \sqrt[n]{\frac{1}{\sqrt{5}^n}\left|\frac{(4n)!}{(n!)^4}\right|} \leq \sqrt[n]{\frac{1}{\sqrt{5}}\left|\frac{(4n)^{4n}}{(n)^{4n}}\right|} = \sqrt[n]{\frac{1}{\sqrt{5}^n}\left|\frac{4^{4n}n^{4n}}{n^{4n}}\right|} = \frac{256}{\sqrt{5}}$$

$$R = \frac{1}{\frac{256}{\sqrt{5}}} = \frac{\sqrt{5}}{256}$$

Frage 3

$$z = (-\sqrt{3}+i)^7 \quad \| \quad z = r^n \cdot e^{in\phi}$$

$$r = \sqrt{\sqrt{3}^2 + 1^2} = 2$$

$$\phi = \arctan\left(\frac{1}{-\sqrt{3}}\right) = -\frac{\pi}{6}$$

$$\Rightarrow z = 2^7 \cdot e^{-i\frac{7\pi}{6}} = 128\, e^{-i\frac{7\pi}{6}}$$

Serie 4

Frage 1

$$z^5 = -\frac{3}{2} + \frac{3^{\frac{3}{2}}}{2} i$$

$$r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3^{\frac{3}{2}}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{27}{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3$$

$$\phi = \arctan\left(\frac{3^{\frac{3}{2}}/2}{-3/2}\right) + \pi = \frac{2\pi}{3} + 2\pi k$$

$$5\phi = \frac{2\pi}{3} + 2\pi k$$

$$\phi = \frac{2\pi}{15} + \frac{2\pi k}{5} \qquad \| \quad \text{mit} \quad 0 \leq k \leq 4$$

$$z_1 = 3^{1/5} \cdot e^{i \cdot \left(\frac{2\pi}{15}\right)}$$

$$z_2 = 3^{1/5} \cdot e^{i \cdot \left(\frac{2\pi}{15} + \frac{2\pi}{5}\right)}$$

$$z_3 = 3^{1/5} \cdot e^{i \left(\frac{2\pi}{15} + \frac{4\pi}{5}\right)}$$

$$z_4 = 3^{1/5} \cdot e^{i \left(\frac{2\pi}{15} + \frac{6\pi}{5}\right)}$$

$$z_5 = 3^{1/5} \cdot e^{i \left(\frac{2\pi}{15} + \frac{8\pi}{5}\right)}$$

Frage 2

$$\text{Log}\left(2^{3/2} + 2^{3/2}i\right) \qquad \| \quad \text{Log}(z) = \log(|z|) + \text{Arg}(z) \cdot i$$

$$|z| = \sqrt{\left(2^{3/2}\right)^2 + \left(2^{3/2}\right)^2} = \sqrt{2^3 + 2^3} = \sqrt{16} = 4$$

$$\text{Arg}(z) = \frac{\pi}{4}$$

$$\Rightarrow \log(4) + \frac{\pi}{4} i$$

Serie 5

Frage 1

$$u(x,y) = 4x^3 y - 4xy^3$$

$$\frac{\partial u}{\partial x} = 12x^2 y - 4y^3 \qquad \Rightarrow \frac{\partial^2 u}{\partial x^2} = 24xy$$

$$\frac{\partial u}{\partial y} = 4x^3 - 12xy^2 \qquad \Rightarrow \frac{\partial^2 u}{\partial y^2} = -24xy$$

$$= 0$$

$$v(x,y) = y^4 - 6x^2 y^2 + x^4$$

$$\Rightarrow \text{Nicht holomorph}$$

$$\frac{\partial v}{\partial x} = -12xy^2 + 4x^3 \qquad \Rightarrow \frac{\partial^2 v}{\partial x^2} = 12x^2$$

$$\frac{\partial v}{\partial y} = 4y^3 - 12x^2 y \qquad = \frac{\partial^2 v}{\partial y^2} = 12y^2$$

$$\neq 0$$

Frage 2

$$f(z) = -z^4 \qquad \| \; z = x + iy$$

$$= -(x+iy)^4 = -(x^4 + 4x^3 iy - 6x^2 y^2 - 4xiy^3 + y^4)$$

$$= -x^4 - 4x^3 iy + 6x^2 y^2 + 4xiy^3 - y^4$$

$$u(x,y) = -x^4 + 6x^2 y^2 - y^4 \quad ; \quad v(x,y) = -4x^3 iy + 4xiy^3$$

$$\frac{\partial u}{\partial x} = -4x^3 + 12xy^2 \qquad \Rightarrow \frac{\partial^2 u}{\partial x^2} = -12x^2 + 12y^2$$

$$\frac{\partial u}{\partial y} = 12x^2 y - 4y^3 \qquad \Rightarrow \frac{\partial^2 u}{\partial y^2} = 12x^2 - 12y^2$$

$$0$$

$$\Rightarrow \text{holomorph}$$

$$\frac{\partial v}{\partial x} = -12x^2 iy + 4iy^3 \qquad \Rightarrow \frac{\partial^2 v}{\partial x^2} = -24xiy$$

$$\frac{\partial v}{\partial y} = -4x^3 i + 12xiy^2 \qquad \Rightarrow \frac{\partial^2 v}{\partial y^2} = 24xiy$$

$$0$$

Frage 3

$$\lim_{z \to 0} \left(\frac{z}{\bar{z}} \right)^2 \qquad \| \; z = x + iy$$

$$\lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{(x+iy)}{(x-iy)} \right)^2 = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 + 2xiy - y^2}{x^2 - 2xiy - y^2} \qquad \| \; \text{Limes von } x \text{ und } y \text{ separat betrachten}$$

$$\lim_{\substack{x \to 0 \\ y = 0}} \frac{x^2}{x^2} = 1^2 = 1$$

$$\lim_{\substack{y \to 0 \\ x = 0}} \frac{-y^2}{-y^2} = 1^2 = 1$$

$$z = x + ix$$

$$\lim_{x \to \infty} \left(\frac{x+ix}{x-ix} \right)^2 = \lim_{x \to \infty} \frac{\cancel{x} + 2ix^2 - \cancel{x}}{\cancel{x^2} - 2ix^2 - \cancel{x}} = -1$$

=> Die Limes sind nicht gleich. Somit existiert der Limes nicht

Frage 5

$$f(z) = \frac{z-4}{z-3} = \frac{x+iy-4}{x+iy-3} = \frac{(x-4)+iy}{(x-3)+iy} \parallel \frac{(x-3)-iy}{(x-3)-iy}$$

$$= \frac{(x-4)(x-3) - iy(x-4) + iy(x-3) + y^2}{(x-3)^2 + y^2}$$

$$\Rightarrow Re(f) = \frac{(x-4)(x-3)+y^2}{(x-3)^2+y^2} \quad ; \quad Im(f) \; \frac{-y(x-4) + y(x-3)}{(x-3)^2+y^2}$$

Frage 1

$$z = t^2$$

$$\frac{dz}{dt} = 2t$$

$$dz = 2t\, dt$$

$$\int_0^2 (t^2)^2 \cdot 2t\, dt = \int_0^2 t^4\, 2t\, dt = 2\int_0^2 t^5\, dt = 2\left[\frac{t^6}{6}\right]_0^2 = \frac{2 \cdot 64}{6} = \frac{128}{6} = \frac{64}{3}$$

Frage 2

$$z_1(t) = 2i + 3 + t(2i + 1 - 2i - 3) = 2i - 2t + 3$$

$$dz = -2\, dt$$

$$z_2(t) = 2i + 1 + t(1 - 2i - 1) = 2i - 2it - 1$$

$$dz = -2i\, dt$$

$$z_3(t) = 1 + t(3 - 1) = 1 + 2t$$

$$dz = 2\, dt$$

$$z_4(t) = 3 + t(2i + 3 - 3) = 3 + 2it$$

$$dz = 2i\, dt$$

$$|z_1|^2 = (3 - 2t)^2 + 2^2 = 13 - 12t + 4t^2$$

$$|z_2|^2 = (2 - 2t)^2 + 1^2 = 5 - 8t + 4t^2$$

$$|z_3|^2 = (1 + 2t)^2 = 1 + 2t + 4t^2$$

$$|z_4|^2 = 2t^2 + 3^2 = 9 + 4t^2$$

$$\int_0^1 (13 - 12t + 4t^2)(-2)\, dt = -2\int_0^1 (13 - 12t + 4t^2)\, dt = -2\left[\frac{4t^3}{3} - 6t^2 + 13t\right]_0^1$$

$$= -2\left[\frac{4}{3} - 6 + 13\right] = -\frac{50}{3}$$

$$\int_0^1 (5 - 8t + 4t^2)(-2i)\, dt = -2i\int_0^1 (5 - 8t + 4t^2)\, dt = -2i\left[\frac{4t^3}{3} - 4t^2 + 5t\right]_0^1$$

$$= -2i\left[\frac{4}{3} - 4 + 5\right] = -\frac{14}{3}i$$

$$\int_0^1 (1+2t+4t^2)(2)\, dt = 2 \cdot \int_0^1 1+2t+4t^2\, dt = 2 \cdot \left[\frac{4t^3}{3} + t^2 + t\right]_0^1$$

$$= 2 \cdot \left[\frac{4}{3} + 1 + 1\right] = \frac{26}{3}$$

$$\int_0^1 (9+4t^2)(2i)\, dt = 2i\int_0^1 9+4t^2\, dt = 2i\left[\frac{4t^3}{3} + 9t\right]_0^1$$

$$2i\left[\frac{4}{3} + 9\right] = \frac{62}{3}i$$

$$\Rightarrow \int_\Gamma |z|^2\, dz = -\frac{24}{3} + \frac{48}{3}i = -8 + 16i$$

Frage 3

$$z(t) = 3+6i + t(5+4i - 3-6i) = 3+6i +2t - 2it$$

$$dz = 2-2i\; dt$$

$$\int_0^1 (3-6i+2t + 2it)(2-2i)\, dt = \int_0^1 6-12i + 4t +4it -6i - 12 -4ti + 4t\; dt$$

$$= \int_0^1 -6 + 8t -18i\, dt = \left[-6t + 4t^2 - 18it\right]_0^1 = -2 - 18i$$

Frage 4

$$z(t) = \frac{\pi}{2}\, e^{it}$$

$$dz = \frac{\pi}{2} i e^{it}\, dt$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi}{2} e^{it}\right)^3 \left(\frac{\pi}{2} i e^{it}\right) dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi^3}{8} e^{3it}\, \frac{\pi}{2}i e^{it}\, dt = \frac{\pi^4}{16}i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{4it}\, dt$$

$$= \frac{\pi^4}{16}i\left[\frac{e^{4it}}{4i}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1 - 1 = 0$$

Frage 5

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{|z - z_0|} = \lim_{z\to z_0} \frac{(z+3)^2 - (z_0+3)^2}{|z-z_0|} = \lim_{z\to z_0} \frac{z^2 + 6z + 9 - z_0^2 - 6z_0 - 9}{z - z_0}$$

$$= \lim_{z\to z_0} \frac{(z-z_0)^2 + 6(z-z_0)}{z-z_0} = \lim_{z\to z_0} \frac{(z+z_0)(z-z_0) + 6(z-z_0)}{z - z_0} = 2z_0 + 6$$

$$f(x,y) = ((x+iy) + 3)^2 = (x+3)^2 + 2iy(x+3) - y^2 = x^2 + 6x + 9 + 2ixy + 6iy - y^2$$

$$= (x^2 - y^2 + 6x + 9) + i(2xy + 6y)$$

$$\frac{\partial f_1}{\partial x} = 2x + 6 + 2yi$$

$$\frac{\partial f_1}{\partial y} = -2y + 2xi + 6i$$

$$u(x,y) = x^2 - y^2 + 6x + 9$$

$$v(x,y) = 2xy + 6y$$

$$\frac{\partial u}{\partial x} = 2x + 6$$

$$\frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial v}{\partial y} = 2x + 6$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \checkmark$$

$\Rightarrow$ Es gilt die Cauchy-Riemann Gleichung

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \checkmark$$

<u>Serie 7</u>

Frage 1

a) $z(t) = -2+2i + 2 \cdot e^{it}$

$2i = -2+2i+2e^{it}$ $-4+2i = -2+2i+2e^{it}$

$e^{it} = 1 \rightarrow \arg(1) = 0$ $e^{it} = -1 \rightarrow \arg(-1) = \pi$

$dz = z'(t)\,dt = 2ie^{it}\,dt$

$\displaystyle\int_\gamma f(z)\,dz = \int_0^\pi \left(4(-2-2i+2e^{-it})+3\right)(2ie^{it})\,dt$

$\displaystyle = \int_0^\pi (-5-8i+8e^{it})(2ie^{it})\,dt$

$\displaystyle = \int_0^\pi -10ie^{it}+16e^{it}+16i\ dt$

$\displaystyle = -10e^{it}-16ie^{it}+16it\ \Big|_0^\pi$

$= -10e^{i\pi}-16ie^{i\pi}+16i\pi+10+16i$

$= 10+16i+16i\pi+10+16i$

$= 20+32i+16i\pi$ i

$\displaystyle\int_\eta f(z)\,dz = \int_0^{-\pi}\left(4(-2-2i+2e^{-it})+3\right)(2ie^{it})\,dt$

$\displaystyle = -10e^{it}-16ie^{it}+16it\ \Big|_{-\pi}^0$

$= -10e^{-i\pi}-16ie^{-i\pi}-16i\pi+10+16i$

$= 10+16i+16i\pi+10+16i$

$= 20+32i+16i\pi$

Frage 2

Frage 3

$z(t) = 0 + 2e^{it}$

$2 = 2e^{it}$ $2i = 2e^{it}$

$e^{it} = 1 \rightarrow \arg(1) = 0$ $e^{it} = i \rightarrow \arg(i) = \frac{\pi}{2}$

$$dz = z'(t)\,dt = 2ie^{it}\,dt$$

Alle drei Integrale haben die gleiche Lösung, da sie den gleichen Weg zurücklegen

$$\int_0^{\pi/2}\left(6(2e^{it})^2 + 4e^{it}\right)(2ie^{it})\,dt$$

$$= \int_0^{\pi/2}\left(24e^{2it} + 4e^{it}\right)(2ie^{it})\,dt$$

$$= \int_0^{\pi/2} 4\,tie^{3it} + 8ie^{2it}\,dt = 48e^{3it} + 8e^{2it}\Big|_0^{\pi/2} = -48i - 8 - 48 - 8 = -64 - 48i$$