Mathematische Methoden (frühere Name: komplexe Analysis) Serien

Jirayu Ruh

Inhaltsverzeichnis

DISCLAIMER

Ich übernehme keine Haftung über mögliche Fehler in den Notizen. Es hat sicherlich ein paar drinnen.

Fehler können per Mail an jirruh@ethz.ch gemeldet werden.

Serie 1

Aufgabe 1

a) i) $6+6i$ ii) $-10+i$ iii) $3-39i$

iv) $\frac{(12+3i)}{(12+6i)} = \frac{(12+3i)(12-6i)}{144+36} = \frac{162-36i}{180}$

b) i) $-i \Rightarrow 1(\cos(-\frac{\pi}{2})+i\sin(-\frac{\pi}{2}))$

$1-i \Rightarrow \sqrt{2}(\cos(-\frac{\pi}{4})+i\sin(-\frac{\pi}{4}))$

$|z| = 1/\sqrt{2} = \frac{\sqrt{2}}{2}$

$\arg(z) = (-\frac{\pi}{2}+\frac{\pi}{4}) = -\frac{\pi}{4}$

$\Rightarrow z = \frac{\sqrt{2}}{2}(\cos(-\pi/4)+i\sin(-\pi/4))$

ii) $-2-2i \Rightarrow \sqrt{8}(\cos(-3\pi/4)+i\sin(-\pi/4))$

$1+\sqrt{3}i \Rightarrow 2(\cos(\pi/3)+i\sin(\pi/3))$

$|z| = \sqrt{8}/2$

$\arg(z) = (-3\pi/4 - \pi/3) = -13\pi/12 = 11\pi/12$

$\Rightarrow z = \frac{\sqrt{8}}{2}(\cos(11\pi/12)+i\sin(11\pi/12))$

iii) $1+\sqrt{3}i \Rightarrow 2(\cos(\pi/3)+i\sin(\pi/3))$

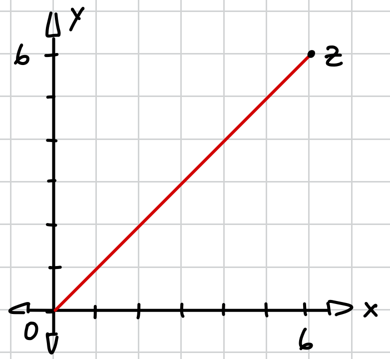
$\sqrt{3}+i \Rightarrow 2(\cos(\pi/6)+i\sin(\pi/6))$

$|z| = 1$

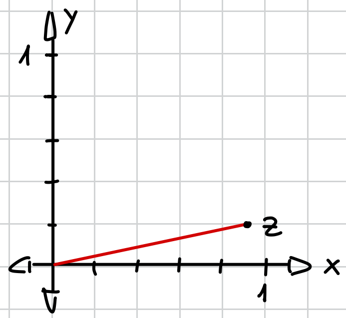
$\arg(z) = (\pi/3-\pi/6) = \pi/6$

$z = 1(\cos(\pi/6)+i\sin(\pi/6))$

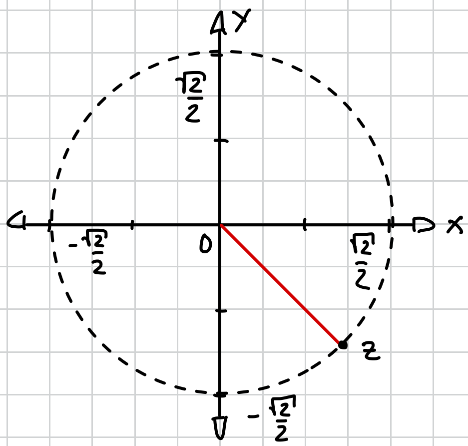
c) ai)



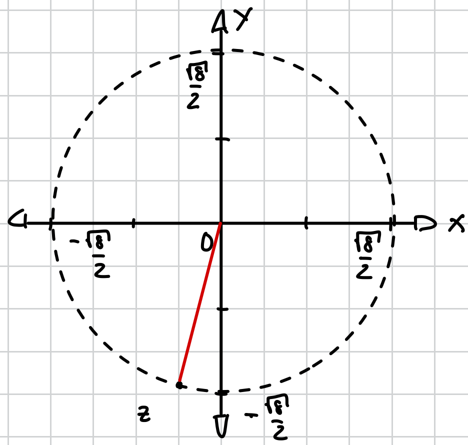
aiv)



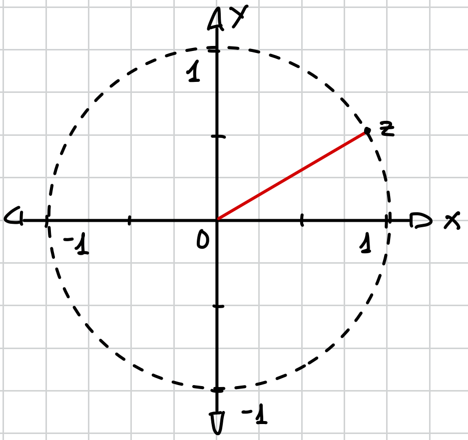
bi)



bii)

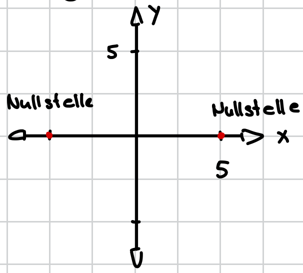


b iii)



Aufgabe 2

i) $z = \pm 5$



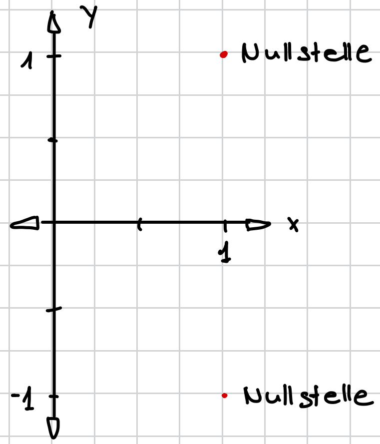
iii) $z^3 + z^2 - 2 : z - 1 = z^2 + 2z + 2$

$$\begin{array}{l} z^3 - z^2 \\ \quad 2z^2 \\ \quad 2z^2 - 2z \\ \qquad 2z - 2 \\ \qquad 2z - 2 \end{array}$$

ii) $z^2 - 2z + 2 \stackrel{!}{=} 0$

$z_{1,2} = 1 \pm \sqrt{1 - 2}$

$= 1 \pm i$

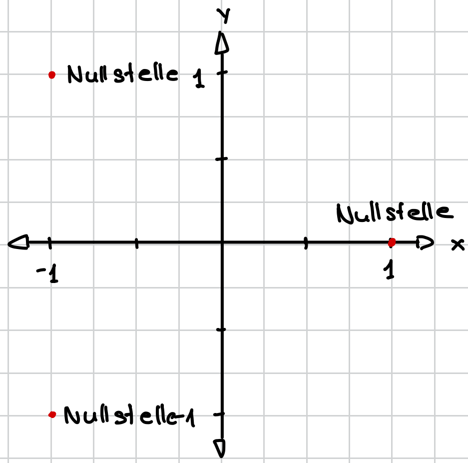


$z_1 = 1$

$z^2 + 2z + 2 \stackrel{!}{=} 0$

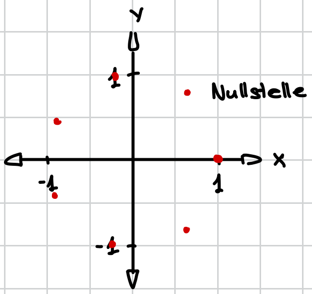
$z_{2,3} = -1 \pm \sqrt{1 - 2}$

$= -1 \pm i$



iv) $z = 1\left(\cos(7\varphi) + i \cdot \sin(7\varphi)\right)$

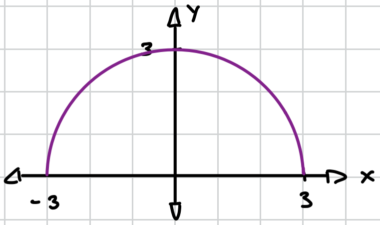
$7\varphi = 2\pi n \Rightarrow \varphi = \frac{2\pi n}{7}$



Serie 2

Aufgabe 1

i) $M = \{ z \in \mathbb{C} \mid |z| = 3,\ \mathrm{Im}(z) \geq 0 \}$



ii) $M = \left\{ z \in \mathbb{C} \;\middle|\; \frac{|z+2-2i|}{|z+i|} = 2 \right\}$

$= \left\{ z \in \mathbb{C} \;\middle|\; \frac{|x+2+i(y-2)|}{|x+i(y+1)|} = 2 \right\}$

$(|x+2+i(y-2)|)^2 = (2|x+i(y+1)|)^2$

$(x+2)^2 + (y-2)^2 = 4(x^2 + (y+1)^2)$

$x^2 + 4x + 4 + y^2 - 4y + \cancel{4} = 4x^2 + 4y^2 + 8y + \cancel{4}$

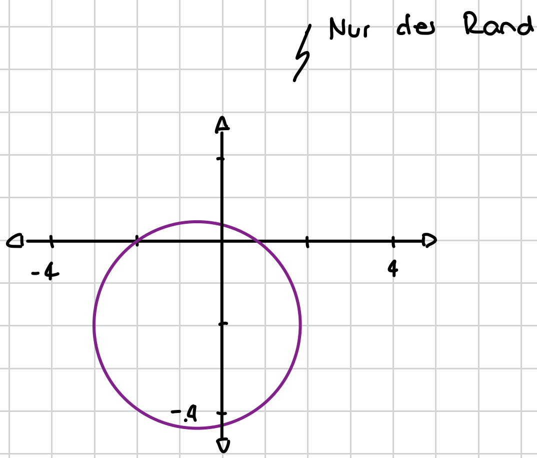
$-3x^2 + 4x + 4 = 3y^2 + 12y$

$3x^2 - 4x + 3y^2 + 12y = 4$

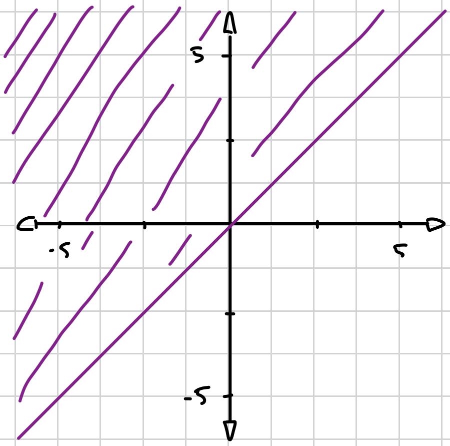
$x^2 - \frac{4}{3}x + y^2 + 4y = \frac{4}{3}$

$(x - \frac{2}{3})^2 + (y+2)^2 = \frac{52}{9}$

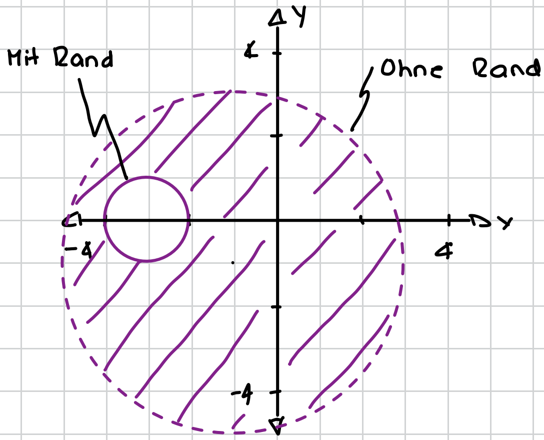
$M = \{ z \in \mathbb{C} \mid |z - \frac{2}{3} + 2i| = \frac{2\sqrt{13}}{3}$



iii) $M = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) \geq \mathrm{Re}(z) \}$



iv) $M = \{ z \in \mathbb{C} \mid |z-3| \geq 1 \text{ und } |z-1-i| < 4 \}$



Serie 3

Aufgabe 1

a) i) $\lim_{|n|\to\infty} \cos(in) = \lim_{|n|\to\infty} \cosh(n) = \underline{\underline{\infty}}$

ii) $\lim_{n\to\infty} 1+(-1)^n \cdot \frac{i}{n} = 1 + \lim_{n\to\infty} (-1^n \cdot \frac{i}{n}) = 1+0i = \underline{\underline{1}}$ || Wenn $n\to\infty$ wird $\frac{i}{n} \to 0$

iii) $\lim_{n\to\infty} \frac{(n+2\pi i)^n}{n^n} = \lim_{n\to\infty} \frac{n^n}{n^n} = \underline{\underline{1}}$ || $2\pi i$ hat kein Einfluss aufs Ergebnis wenn $n\to\infty$

iv) $\lim_{n\to\infty} \operatorname{Arg}(1+(-1)^n \frac{i}{n})$ || $\lim_{n\to\infty} 1+(-1)^n \frac{i}{n} = 1$

$\operatorname{Arg}(1) = \underline{\underline{0}}$

b)

$$\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(\pi i)^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{2^n} + \sum_{n=0}^{\infty} \frac{(\pi i)^n}{n!} = 2 + \exp(\pi i) = \underline{\underline{2 + e^{\pi i}}}$$

Aufgabe 2

$\exp(it) = \cos(t) + i \cdot \sin(t)$

$\exp(z_1 + z_2) = \exp(z_1) \cdot \exp(z_2)$ || $z_1 = ix$; $z_2 = iy$

$\exp(i(x+y)) = \exp(ix) \cdot \exp(iy)$ || Eulerformel

$\exp(i(x+y)) = (\cos(x) + i\sin(x)) \cdot (\cos(y) + i\sin(y))$

$\cos(x+y) + i\sin(x+y) = (\cos(x) + i\sin(x)) \cdot (\cos(y) + i\sin(y))$

$\cos(x+y) + i\sin(x+y) = \cos(x)\cos(y) + \cos(x)i\sin(y) + i\sin(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x+y) + i\sin(x+y) = (\cos(x)\cos(y) - \sin(x)\sin(y)) + i(\cos(x)\sin(y) + \cos(y)\sin(x))$

$\Rightarrow \cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\Rightarrow \sin(x+y) = \cos(x)\sin(y) + \cos(y)\sin(x)$

|| Ich bin zu faul um es für x-iy zu beweisen :-)

$\square$

Aufgabe 3

i) $\lim_{z\to 0} \frac{\bar{z} + z^2}{z}$ || $z = x+iy$

$\lim_{z\to 0} \frac{x - iy + x^2 + 2xiy - y}{x+iy}$ || Wir müssen den Grenzwert von Re und Im seperat betrachten

$\lim_{y\to 0} \frac{x - iy + x^2 + 2xiy - y^2}{x+iy}$ || $x=0$

$\lim_{y\to 0} \frac{-iy - y^2}{iy} = \lim_{y\to 0} -1 - \frac{y^2}{iy} = \lim_{y\to 0} -1 - \frac{y}{i} = \underline{\underline{-1}}$

Aufgabe 3

$$\lim_{x \to 0} \frac{x - iy + x^2 + 2xiy - y^2}{x + iy} \quad \Big\| \; y = 0$$

$$\lim_{x \to 0} \frac{x + x^2}{x} = \lim_{x \to 0} 1 + x = 1$$

$z_0 = 1 - i \neq 0$ ‖ Somit existiert des Limes nicht.

ii) $$\lim_{z \to 0} \frac{\cos(z) - 1}{z^2} \quad \Big\| \; \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \quad \text{(Taylorreihe)}$$

$$\lim_{z \to 0} \frac{-\frac{z^2}{2!} + \frac{z^4}{4!} + \dots}{z^2} = \underline{\underline{-\frac{1}{2}}}$$

iii) $$\lim_{z \to 0} \frac{\sin(z)}{\bar{z}} \quad \Big\| \; z = x + iy$$

$$\lim_{z \to 0} \frac{\sin(x + iy)}{x - iy}$$ ‖ Wir müssen den Grenzwert von Re und Im seperat betrachten

$$\lim_{x \to 0} \frac{\sin(x + iy)}{x - iy} \quad \Big\| \; y = 0$$

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad \Big\| \; \lim_{x \to 0} \frac{\sin(x)'}{x'} = \frac{\cos(x)}{1} = 1 \quad \text{(Bernoulli - l'Hôpital)}$$

$$\lim_{y \to 0} \frac{\sin(x + iy)}{x - iy} \quad \Big\| \; x = 0$$

$$\lim_{y \to 0} \frac{\sin(iy)}{-iy} = \lim_{y \to 0} \frac{\cancel{i} \cdot \sin(y)}{-\cancel{i}y} = \lim_{y \to 0} -\frac{\sin(y)}{y} = -1 \quad \Big\| \; \text{Bernoulli - l'Hôpital}$$

$\Rightarrow z_0 = 1 - i \neq 0$ ‖ Somit existiert der Limes nicht.

Serie 4

Aufgabe 1

a) i) $e^{i} = \cos(1) + i\sin(1)$

ii) $e^{1-2i} = e \cdot e^{-2i}$

$e^{-2i} = \cos(-2) + i\sin(-2)$

$\Rightarrow e^{1-2i} = e\cos(-2) + e \cdot i\sin(-2)$

iii) $\operatorname{Log}(1+i) = \log(\sqrt{2}) + i\frac{\pi}{4}$

b) i) $\cos(10i) = \cosh(10)$

$\cosh(10) = \dfrac{e^{x} + e^{-x}}{2}$

ii) $\sin(5+5i) = \sin(5)\cos(5i) + \cos(5)\sin(5i)$

$\cos(5i) = \cosh(5) = \dfrac{e^{5}+e^{-5}}{2}$

$\sin(5i) = \sinh(5) = \dfrac{e^{5}-e^{-5}}{2}$

$\sin(5+5i) = \sin(5)\dfrac{e^{5}+e^{-5}}{2} + \cos(5)\dfrac{e^{5}-e^{-5}}{2}$

iii) $\sin(2-i) = \sin(2)\cos(i) - \cos(2)\sin(i)$

$\cos(i) = \dfrac{e^{1}+e^{-1}}{2}$

$\sin(i) = \dfrac{e^{1}-e^{-1}}{2}$

$\sin(2-i) = \sin(2)\dfrac{e^{1}+e^{-1}}{2} - \cos(2)\dfrac{e^{1}-e^{-1}}{2}$

Aufgabe 2

$\operatorname{Log}(z_1 \cdot z_2) \neq \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$

$\Rightarrow \begin{cases} z_1 = -1 \\ z_2 = i \end{cases}$

// Wir suchen z_1 und z_2, sodass die Summe der Arg() > π sind aber das Arg() von $z_1 \cdot z_2$ kleiner als π ist.

Aufgabe 3

$(\cos(\phi) + i\sin(\phi))^n = \cos(n\phi) + i \cdot \sin(n\phi)$ || Beweis durch Induktion

$n = 0$

$(\cos(\phi) + i\sin(\phi))^n = \cos(n\phi) + i \cdot \sin(n\phi)$

$1 = 1 + 0$ ✓

$n = 1$

$(\cos(\phi) + i\sin(\phi))^1 = \cos(\phi) + i\sin(\phi)$ ✓

$n = k+1$

$(\cos(\phi) + i\sin(\phi))^{k+1} = (\cos(\phi) + i\sin(\phi))^k \cdot (\cos(\phi) + i\sin(\phi))$

$= (\cos(k\phi) + i\sin(k\phi))(\cos(\phi) + i\sin(\phi))$

$= \cos(k\phi)\cos(\phi) + \cos(k\phi) i \sin(\phi) + i\sin(k\phi)\cos(\phi) - \sin(k\phi)\sin(\phi)$ || Kosinus & Sinus Satz

$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$\sin(A+B) = \cos(A)\sin(B) + \cos(B)\sin(A)$

$\Rightarrow \cos(k\phi + k) + \sin(k\phi + k) = \cos((k+1)\phi) + \sin((k+1)\phi)$

$\square$

Serie 5

Aufgabe 1

$$f'(z) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} \quad \Big\| \; \Delta z = z - z_0$$

i) $f(z) = 3z^3 + z - 3$

$$f'(z) = \lim_{\Delta z\to 0} \frac{3(z+\Delta z)^3 + (z+\Delta z) - 3 - (3z^3+z-3)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{3(z^3 + 3z^2\Delta z + 3z\Delta z^2 + \Delta z^3) + (z+\Delta z) - 3 - (3z^3+z-3)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{\cancel{3z^3} + 9z^2\Delta z + 9z\Delta z^2 + 3\Delta z^3 + \cancel{z} + \Delta z \cancel{-3} - \cancel{3z^3} - \cancel{z} \cancel{+3}}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{9z^2\Delta z + 9z\Delta z^2 + 3\Delta z^3 + \Delta z}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{\cancel{\Delta z}(9z^2 + 9z\Delta z + 3\Delta z^2 + 1)}{\cancel{\Delta z}}$$

$$= \lim_{\Delta z\to 0} 9z^2 + 9z\Delta z + 3\Delta z^2 + 1$$

$$\Rightarrow \underline{\underline{f'(z) = 9z^2 + 1}}$$

ii) $f(z) = \sin(\mathrm{Re}(z)) = \sin(x)$ || Grundsätzlich wäre diese Gleichung ableitbar aber es ist in einem Bereich ableitbar.

$f'(z) = \cos(x)$

$\tilde{x} \in [\pi/2 + k\pi]$

$\Rightarrow$ Wenn $x \in \tilde{x}$ dann existiert die Ableitung, sonst nicht

iii) $f(z) = \dfrac{1}{z^2}$

$$f'(z) = \lim_{\Delta z\to 0} \frac{\frac{1}{(z+\Delta z)^2} - \frac{1}{z^2}}{\Delta z} = \lim_{\Delta z\to 0} \frac{\frac{z^2-(z+\Delta z)^2}{z^2(z+\Delta z)^2}}{\Delta z} = \lim_{\Delta z\to 0} \frac{z^2-(z+\Delta z)^2}{\Delta z\,(z^2(z+\Delta z)^2)}$$

$$= \lim_{\Delta z\to 0} \frac{\cancel{z^2} - \cancel{z^2} - 2z\Delta z - \Delta z^2}{\Delta z\,(z^2(z+\Delta z)^2)} = \lim_{\Delta z\to 0} \frac{\cancel{\Delta z}(-2z-\Delta z)}{\cancel{\Delta z}\,(z^2(z+\Delta z)^2)}$$

$$\Rightarrow \underline{\underline{f'(z) = \frac{-2z}{z^4} = -\frac{2}{z^3}}}$$

iv) $f(z) = e^{-\pi z^2}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{e^{-\pi (z+\Delta z)^2} - e^{-\pi z^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{e^{-\pi (z^2 + 2z\Delta z + \Delta z^2)} - e^{-\pi z^2}}{\Delta z} = \lim_{\Delta z \to 0} \frac{e^{-\pi z^2} \cdot e^{-\pi (2z\Delta z + \Delta z^2)} - e^{-\pi z^2}}{\Delta z}$$

$\Rightarrow$ Wir verwenden die Aproximation, dass $e^{-\pi(2z\Delta z + \Delta z^2)} \approx 1 - \pi(2z\Delta z + \Delta z^2)$ für $\Delta z \to 0$ (Taylor Series $\to e^x = 1 + x + \frac{x^2}{2!} + \ldots$)

$$= \lim_{\Delta z \to 0} \frac{e^{-\pi z^2} (\cancel{1} - \pi (2z\Delta z + \Delta z^2) - \cancel{1})}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{e^{-\pi z^2} (-\pi (2z \cancel{\Delta z} + \Delta z^{\cancel{2}}))}{\cancel{\Delta z}}$$

$$\Rightarrow f'(z) = -2\pi z \cdot e^{-\pi z^2}$$

Aufgabe 2

$g(t) = f(\gamma(t))$

$$g'(t) = \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(\gamma(t+\Delta t)) - f(\gamma(t))}{\Delta t} \quad \Bigg\| \cdot \frac{\gamma(t+\Delta t) - \gamma(t)}{\gamma(t+\Delta t) - \gamma(t)}$$

$$= \lim_{\gamma(t+\Delta t) \to \gamma(t)} \frac{f(\gamma(t+\Delta t)) - f(\gamma(t))}{\gamma(t+\Delta t) - \gamma(t)} \cdot \lim_{\Delta t \to 0} \frac{\gamma(t+\Delta t) - \gamma(t)}{\Delta t} \quad \Bigg\| \; \Delta\gamma = \gamma(t+\Delta t) - \gamma(t) \Rightarrow \Delta z = z - z_0$$

$$\Rightarrow g'(t) = f'(\gamma(t)) \cdot \dot{\gamma}(t)$$

Aufgabe 3

Cauchy-Riemann Gleichung: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

a) $u(x,y) := \sin(x^2 - y^2) \cosh(2xy)$

$$\frac{\partial u}{\partial x} = 2x \cdot \cos(x^2 - y^2) \cosh(2xy) + 2y \cdot \sin(x^2 - y^2) \sinh(2xy)$$

$$\frac{\partial u}{\partial y} = -2y \cdot \cos(x^2 - y^2) \cosh(2xy) + 2x \cdot \cos(x^2 - y^2) \sinh(2xy)$$

$$v(x,y) = -\cos(x^2 - y^2) \sinh(2xy)$$

$$\frac{\partial v}{\partial x} = 2x \cdot \sin(x^2 - y^2) \sinh(2xy) - 2y \cos(x^2 - y^2) \cosh(2xy)$$

$$\frac{\partial v}{\partial y} = -2y \cdot \sin(x^2 - y^2) \sinh(2xy) - 2x \cdot \cos(x^2 - y^2) \cosh(2xy)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \checkmark$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \checkmark$$

$\Rightarrow$ Da die Cauchy-Riemann Gleichungen gelten, ist die Gleichung holomorph

b) $u(x,y) := e^{x^2-y^2} \cos(2xy)$

$$\frac{\partial u}{\partial x} = 2x \cdot e^{x^2-y^2} \cdot \cos(2xy) - 2y \cdot e^{x^2-y^2} \cdot \sin(2xy)$$

$$\frac{\partial u}{\partial y} = -2y \cdot e^{x^2-y^2} \cdot \cos(2xy) - 2x \cdot e^{x^2-y^2} \cdot \sin(2xy)$$

$v(x,y) := e^{x^2-y^2} \sin(2xy)$

$$\frac{\partial v}{\partial x} = 2x\, e^{x^2-y^2} \cdot \sin(2xy) + 2y \cdot e^{x^2-y^2} \cdot \cos(2xy)$$

$$\frac{\partial v}{\partial y} = -2y \cdot e^{x^2-y^2} \cdot \sin(2xy) + 2x \cdot e^{x^2-y^2} \cdot \cos(2xy)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \checkmark$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \checkmark$$

$\Rightarrow$ Da die Cauchy-Rieman Gleichung gelten, Ist die Gleichung holomorph

---

Aufgabe 4

$x = r \cdot \sin(\varphi)$

$y = r \cdot \cos(\varphi)$

$r = \sqrt{x^2+y^2}$

$\varphi = \arctan\left(\frac{x}{y}\right)$

$\tilde{u}(x,y) = \tilde{u}(r \cdot \sin(\varphi), r \cdot \cos(\varphi))$

$\tilde{v}(x,y) = \tilde{v}(r \cdot \sin(\varphi), r \cdot \cos(\varphi))$

$$\frac{\partial \tilde{u}}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial r}$$ Kettenregel

$$\frac{\partial \tilde{u}}{\partial \varphi} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial \varphi}$$

$$\frac{\partial \tilde{v}}{\partial r} = \frac{\partial v}{\partial x} \frac{\partial v}{\partial r} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial r}$$

$$\frac{\partial \tilde{v}}{\partial \varphi} = \frac{\partial v}{\partial x} \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial \varphi}$$

$$\frac{\partial \tilde{u}}{\partial r} = \frac{\partial u}{\partial x} \cdot \cos(\varphi) + \frac{\partial u}{\partial y} \cdot \sin(\varphi) \Rightarrow \frac{\partial v}{\partial y} \cdot \cos(\varphi) - \frac{\partial v}{\partial x} \cdot \sin(\varphi)$$

$$\frac{\partial \tilde{u}}{\partial \varphi} = \frac{\partial u}{\partial x} \cdot -r \cdot \sin(\varphi) + \frac{\partial u}{\partial y} \cdot r \cdot \cos(\varphi) \Rightarrow \frac{\partial v}{\partial y} \cdot -r \cdot \sin(\varphi) - \frac{\partial v}{\partial x} \cdot r \cdot \cos(\varphi)$$

$$\frac{\partial \tilde{v}}{\partial r} = \frac{\partial v}{\partial x} \cdot \cos(\varphi) + \frac{\partial v}{\partial y} \cdot \sin(\varphi) \Rightarrow -\frac{\partial u}{\partial y} \cdot \cos(\varphi) + \frac{\partial u}{\partial x} \cdot \sin(\varphi)$$

$$\frac{\partial \tilde{v}}{\partial \varphi} = \frac{\partial v}{\partial x} \cdot -r \cdot \sin(\varphi) + \frac{\partial v}{\partial y} \cdot r \cdot \cos(\varphi) \Rightarrow -\frac{\partial u}{\partial y} \cdot -r \cdot \sin(\varphi) + \frac{\partial u}{\partial x} \cdot r \cdot \cos(\varphi)$$

$$r \cdot \frac{\partial \hat{u}}{\partial r} = r \cdot \left(\frac{\partial v}{\partial y} \cdot \cos(\varphi) - \frac{\partial v}{\partial x} \cdot \sin(\varphi) \right) = \frac{\partial v}{\partial y} \cdot r \cdot \cos(\varphi) - \frac{\partial v}{\partial x} \cdot r \cdot \sin(\varphi) = \frac{\partial \tilde{v}}{\partial r} \checkmark$$

$$-r \cdot \frac{\partial \tilde{v}}{\partial r} = -r \left(-\frac{\partial u}{\partial y} \cdot \cos(\varphi) + \frac{\partial u}{\partial x} \cdot \sin(\varphi) \right) = \frac{\partial u}{\partial y} \cdot r \cdot \cos(\varphi) - \frac{\partial u}{\partial x} \cdot \sin(\varphi) = \frac{\partial \tilde{u}}{\partial \varphi} \checkmark$$

$\square$

Serie 6

Aufgabe 1

i) $f(t) = z_0 + t(z_1 - z_0)$

$= i + t(4+4i)$

$\Rightarrow f(t) = 4t + i(1+4t)$

ii) $f(t) = z_0 + r \cdot e^{it}$

$= 3-2i + 5(\cos(t) + i \cdot \sin(t))$

$\Rightarrow f(t) = 3 + 5\cos(t) + i(\sin(t) - 2)$

Aufgabe 2

i) $\int_\gamma f(z) + g(z)\, dz = \int_0^1 (f(\gamma(t)) + g(\gamma(t))) \cdot \gamma'(t)\, dt$ || Summenregel für Integrale: $\int f(x)+g(x)\,dx = \int f(x)dx + \int g(x)dx$

$= \int_0^1 f(\gamma(t)) \cdot \gamma'(t)\, dt + \int_0^1 g(\gamma(t)) \cdot \gamma'(t)\, dt = \int_\gamma f(z)\, dz + \int_\gamma g(z)\, dz$ $\square$

ii) $|\gamma| := \int_0^1 |\dot{\gamma}(t)|\, dt$

$\left|\int_\gamma f(z)\, dz\right| = \left|\int_0^1 f(\gamma(t)) \cdot \gamma'(t)\, dt\right| \leq \int_0^1 |f(\gamma(t)) \cdot \gamma'(t)|\, dt = \int_0^1 |f(\gamma(t))| \cdot |\gamma'(t)|\, dt$

Sei $|f(\gamma(t))|$ maximal $(\max_{t\in[0,1]} |f(\gamma(t))|$

$\Rightarrow \max_{t\in[0,1]} |f(\gamma(t))| \cdot \int_0^1 \gamma'(t)\, dt = \max_{t\in[0,1]} \cdot \gamma$

Und es gilt auch $\max_{t\in[0,1]} |f(\gamma(t))| = \max_{z\in\gamma([0,1])} |f(z)|$

(Beide Aussagen sind äquivalent) $\square$

iii) $\overline{\int_\gamma f(z)\, dz} = \overline{\int_0^1 f(e^{2\pi i t}) \cdot 2\pi i e^{2\pi i t}\, dt} = \int_0^1 \overline{f(e^{2\pi i t}) \cdot 2\pi i e^{2\pi i t}}\, dt$

$= \int_0^1 \overline{f(e^{2\pi i t})} \cdot (-2\pi i e^{-2\pi i t})\, dt$ || $\overline{2\pi i} = -2\pi i$ und $\overline{e^{2\pi i t}} = \overline{\cos(2\pi t) + i\sin(2\pi t)}$ $= \cos(2\pi t) - i\sin(2\pi t) = \cos(-2\pi t) + i\sin(-2\pi t)$

$= -\int_0^1 \overline{f(z)} \cdot (2\pi i e^{-2\pi i t})\, dt$ || $e^{-2\pi i t} = \frac{e^{2\pi i t}}{e^{4\pi i t}}$

$= -\int_0^1 \frac{\overline{f(z)}}{z^2} \cdot 2\pi i e^{2\pi i t}\, dt = \int_\gamma \frac{\overline{f(z)}}{z^2}\, dz$ $\square$

Aufgabe 3

$$\int_\gamma \operatorname{Im}(z)\,dz = \int_0^{2\pi} \operatorname{Im}(z)\, z'(t)dt = \int_0^{2\pi} \sin(t)\cdot(-\sin(t)+i\cos(t))\,dt = \int_0^{2\pi} -\sin(t)^2 + i\sin(t)\cos(t)dt$$

$$\Downarrow$$

$$\frac{dz}{dt} = z'(t)$$

$$\Rightarrow \int_0^{2\pi} -\sin(t)^2\,dt = -\int_0^{2\pi} \frac{1-\cos(t)}{2}\,dt = -\left[\frac{t}{2} - \frac{\sin(2t)}{4}\right]_0^{2\pi} = -\pi$$

$$\Rightarrow \int_0^{2\pi} i\cdot\sin(t)\cdot\cos(t)\,dt = i\int_0^{2\pi}\frac{1}{2}\cdot\sin(2t)\,dt = \frac{i}{2}\left[-\frac{\cos(2t)}{2}\right]_0^{2\pi} = 0$$

$$\Rightarrow -\pi + 0 = \underline{\underline{-\pi}}$$

Aufgabe 4

$$\int_\gamma \frac{1}{z^n}\,dz \quad \Big\| \quad z(t) = \cos(t) - i\cdot\sin(t) = e^{-it} \text{ (Uhrzeigersinn)}$$

$$= \int_0^{2\pi} \frac{1}{z^n}\cdot z'(t)\,dt = \int_0^{2\pi} e^{int}\cdot -i\cdot e^{-it}\,dt = \int_0^{2\pi} -i\cdot e^{int-it}\,dt = \int_0^{2\pi} -i\cdot e^{(n-1)it}\,dt$$

$\Rightarrow n = 1$

$$\rightarrow \int_0^{2\pi} -i\,dt = [-it]_0^{2\pi} = -i2\pi$$

$\Rightarrow n \neq 1$

$$\rightarrow \int_0^{2\pi} -i\cdot e^{(n-1)it} = -i\left[\frac{e^{(n-1)it}}{(n-1)i}\right]_0^{2\pi} = 1-1 = 0 \quad \Big\| \quad e^{i2\pi(n-1)} = \cos((n-1)2\pi) + i\sin((n-1)2\pi)$$

$$\Rightarrow -i2\pi - 0 = \underline{\underline{-i2\pi}}$$

Serie 7

Aufgabe 1

$$\int_\gamma \cos\left(\frac{z}{2}\right) dz = 0$$ || Das Wegintegral ist null, da ein geschlossener Weg ist.

Aufgabe 2

a) $$\int z^n \, dz = \frac{z^{(n+1)}}{n+1} + c$$

Für $n=-1$

$$z^n = \frac{1}{z}$$

$$\int \frac{1}{z} dz = \mathrm{Log}(z)$$ || $\mathrm{Log}(z) \in (-\infty, 0]$ welches nicht im Bereich $\mathbb{C} \setminus \{0\}$

$$\Rightarrow \begin{cases} \dfrac{z^{(n+1)}}{n+1} & \text{für } n \neq -1 \\ \text{existiert nicht} & \text{für } n=-1 \end{cases}$$

b) Siehe a) für Rechenweg

$$\Rightarrow \begin{cases} \dfrac{z^{(n+1)}}{n+1} & \text{für } n \neq -1 \\ \mathrm{Log}(z) & \text{für } n=-1 \end{cases}$$

c) $$\exp(\ell(z)) = z$$

$$\exp(\ell(z)) \cdot \ell'(z) = 1$$ || $\exp(\ell(z)) = z$

$$\ell'(z) = \frac{1}{z}$$ || von a) wissen wir, dass

$\square$

Aufgabe 3

a) Von CRDG wissen wir, dass $\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y)$

$$\Rightarrow z = x+iy$$

$$f(z) = u(x,y) + iv(x,y)$$

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i \frac{\partial v}{\partial x}(x,y)$$ || CRDG

$$\underline{\underline{f'(z) = \frac{\partial u}{\partial x}(x,y) - i \frac{\partial u}{\partial y}(x,y)}}$$

b) Wir nehmen an, dass $g(z) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y) = P(x,y) + Q(x,y)$

$$\left.\begin{aligned} \frac{\partial P}{\partial x}(x,y) &= \frac{\partial Q}{\partial y}(x,y) \\ \frac{\partial P}{\partial y}(x,y) &= -\frac{\partial Q}{\partial x}(x,y) \end{aligned}\right\} \text{CRDG}$$

$$\frac{\partial P}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial Q}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \Big\| \quad \text{Gegeben ist } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} \quad \checkmark$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\frac{\partial Q}{\partial x} = -\frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \Big\| \quad \text{Wir nehmen an, dass die Funktion stetig ist}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y} \quad \checkmark$$

c) f ist holomorph $\rightarrow$ Es existiert eine Ableitung g, so dass $f(z) \int g(z)\,dz$

$$f'(z) = g(z)$$

$$\frac{\partial u_1}{\partial x} - i\frac{\partial u_1}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} \rightarrow \int \frac{\partial u_1}{\partial x} = \int \frac{\partial u}{\partial x} \rightarrow u_1(x,y) = u(x,y) + c$$

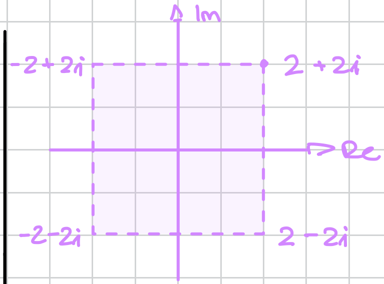
Wenn $c = 0$ ist, so gilt $u_1(x,y) = u(x,y)$

$$\Rightarrow \operatorname{Re}(f(z)) = u(x,y) \rightarrow \operatorname{Re}(f) = u$$

d) Nein, da $i\int \frac{\partial v}{\partial y} = iv(x,y) + iC$ wobei C beliebig gewählt werden kann.

Serie 8

Aufgabe 1



$\frac{z}{2z+1}$ $\quad z = -\frac{1}{2}$ => damit der Nenner 0 wird

$$\operatorname{Res}\left(f(z), -\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right) \cdot \frac{z}{2z+1} = \lim_{z\to -\frac{1}{2}} \cancel{\left(z+\frac{1}{2}\right)} \frac{z}{2\cancel{(z+\frac{1}{2})}} = \frac{z}{2} = -\frac{1}{4}$$

$$\int_\gamma \frac{z}{2z+1}\,dz = 2\pi i \cdot \operatorname{Res}\left(f(z), -\frac{1}{2}\right) = 2\pi i \cdot -\frac{1}{4} = \underline{\underline{-\frac{\pi i}{2}}}$$

|| Residum erst in Kapitel 4 :-)

Aufgabe 2

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)}\,dz$$

$$f^2(s) = \frac{n!}{2\pi i} \int_\gamma \frac{f(s)}{(s-z)^3} = \frac{2!}{2\pi i} \int_\gamma \frac{s^3+2s}{(s-z)^3}\,ds$$

$\Rightarrow f(s) = s^3+2s$

$f^1(s) = 3s^2+2$

$f^2(s) = 6s$ || Wenn wir $s=z$ setzen, so erhalten wir $6z$

$$6z = \frac{1}{\pi i} \int_\gamma \frac{s^3+2s}{(s-z)^3}\,ds$$

$\Rightarrow 6z\pi i = \int_\gamma \frac{s^3+2s}{(s-z)^3}\,ds$ wenn $g(z)$ innerhalb von C ist.

Die Funktion $f(s) = s^3+2s$ ist stetig und wohldefiniert.

$\int_\gamma \frac{s^3+2s}{(s-z)^3}\,ds = 0$ wenn $g(z)$ ausserhalb von C ist.

Aufgabe 3

$f^2(z) = 0$ für $z \in \mathbb{C}$

$$f^2(z) = \frac{2!}{2\pi i} \int_{C_R} \frac{f(w)}{(w-z)^3}\,dw = \frac{1}{\pi i} \int_{C_R} \frac{f(w)}{(w-z)^3}$$

|| $C_R = z + Re^{i\theta}$ mit $0 \leq \theta \leq 2\pi$

$dw = iRe^{i\theta}\,d\theta$

$$f^{(2)}(z) = \frac{1}{\pi i} \int_0^{2\pi} \frac{f(z+Re^{i\theta})}{(Re^{i\theta})^2} iRe^{i\theta} d\theta$$

$$= \frac{1}{\pi R^2} \int_0^{2\pi} f(z+Re^{i\theta}) e^{-2i\theta} d\theta$$

Wir wissen, dass $|f(z)| \leq A|z| + B$

$$|f(z+Re^{i\theta})| = A|z+Re^{i\theta}| + B \leq A(|z|+R)+B$$

$$|f^{(2)}(z)| = \left| \frac{1}{\pi R^2} \int_0^{2\pi} f(z+Re^{i\theta}) \; e^{-2i\theta} \, d\theta \right| \leq \frac{1}{\pi R^2} \int_0^{2\pi} |f(z+Re^{i\theta})| |e^{-2i\theta}| d\theta \quad \| \; |e^{-2i\theta}| = 1$$

$$= \frac{1}{\pi R^2} \int_0^{2\pi} (A(|z|+R)+B \, d\theta = \frac{1}{\pi R^2} (A(|z|+R)+B)2\pi = \frac{2(A(|z|+AR+B)}{R^2}$$

Wenn $R \to \infty$ $|f^{(2)}(0)| = 0$

$f^{(2)}(z) = 0$

$f^{(1)}(z) = a$

$f(z) = az + b$

$\square$

Aufgabe 4

$$\int_\gamma f(z) \, dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

Für $z = 1$

$A = -1$

Für $z = 2$

$B = 1$

$$\Rightarrow \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\mathrm{Res}(f,1) = \lim_{z \to 1} \cancel{(z-1)} \frac{1}{\cancel{(z-1)}(z-2)} = -1$$

$$\mathrm{Res}(f,2) = \lim_{z \to 2} \cancel{(z-2)} \frac{1}{(z-1)\cancel{(z-2)}} = 1$$

a) $\int_{\gamma_1} f(z)\,dz = 2\pi i \cdot \mathrm{Res}(f, z=1) = -2\pi i$

$\int_{\gamma_2} f(z)\,dz = 2\pi i \cdot \mathrm{Res}(f, z=2) = 2\pi i$

b) $\int_{\gamma_3} f(z)\,dz = 2\pi i \cdot (\mathrm{Res}(f, z=1) - \mathrm{Res}(f, z=2)) = 4\pi i$

c) $\int_{\gamma_4} f(z)\,dz = 2\pi i \cdot (\mathrm{Res}(f, z=1) + \mathrm{Res}(f, z=2)) = 0$

d) $\int_{\gamma_5} f(z)\,dz = 2\pi i \cdot (\mathrm{Res}(f, z=1) - \mathrm{Res}(f, z=2)) = 4\pi i$

DISCLAIMER

Die Notizen zu den Stack Aufgaben sind auf meine Werte angepasst. Die Werte können von deinen Aufgaben abweichen.

Serie 1

Frage 1

$$(8-i)(5i+3) = 40i + 24 + 5 - 3i = 37i + 29$$

$$i\,(5i+3) = -5 + 3i$$

$$\bar{w} = -5i + 3$$

$$\frac{8-i}{5i+3} = \frac{(8-i)(5i-3)}{-34} = \frac{40i - 24 + 5 + 3i}{-34} = -\frac{43i}{34} + \frac{19}{34}$$

$$|z| \quad \sqrt{8^2 + 1^2} = \sqrt{65}$$

Frage 2

$$\cos\left(-\tfrac{\pi}{4}\right) > 0 \,;\ \cos\left(\tfrac{\pi}{4}\right) > 0 \quad \Rightarrow \quad Re(z) > 0$$

$$\sin\left(-\tfrac{\pi}{4}\right) < 0 \,;\ \sin\left(\tfrac{\pi}{4}\right) > 0 \quad \Rightarrow \quad Im(z) >/=/< 0$$

Frage 3

$$(4+2i)(4+2i) = 16 + 8i + 8i - 4 = 12 + 16i$$

$$(12+16i)(4+2i) = 48 + 24i + 64i - 32 = 16 + 88i$$

Bemerkung: Teilweise ist es einfacher potenzierte komplexe Gleichungen in Polarform zu berechnen.

Serie 2

Serie 3

Frage 1

$Im(z) = 0$ ‖ Des Im von z muss 0 sein damit $z \in \mathbb{R}$

$$z = e^{\frac{3\pi}{4}i} \cdot (\sqrt{2} + bi)$$

$$= \left(\cos\left(\tfrac{3\pi}{4}\right) + i \sin\left(\tfrac{3\pi}{4}\right)\right) \cdot (\sqrt{2} + bi)$$

$$= \sqrt{2}\cos\left(\tfrac{3\pi}{4}\right) + bi\cos\left(\tfrac{3\pi}{4}\right) + \sqrt{2}\, i \sin\left(\tfrac{3\pi}{4}\right) - b\sin\left(\tfrac{3\pi}{4}\right) \quad \| \, Im(z) = 0$$

$$\Rightarrow bi\cos\left(\tfrac{3\pi}{4}\right) + \sqrt{2}\, i \sin\left(\tfrac{3\pi}{4}\right) = 0$$

$$bi\cos\left(\tfrac{3\pi}{4}\right) = -\sqrt{2}\, i \sin\left(\tfrac{3\pi}{4}\right)$$

$$\cancel{-}\, b\cancel{i}\, \cancel{\tfrac{\sqrt{2}}{2}} = \cancel{-i}$$

$$b = \frac{2\sqrt{2}}{2}$$

Frage 2

$$c_n = \frac{(4n)!}{(-2-i)^n \cdot (n!)^4}$$

$$R = \frac{1}{\limsup\limits_{n \to \infty} \sqrt[n]{|c_n|}}$$

$$\limsup_{n\to\infty} \sqrt[n]{\left| \frac{(4n)!}{(-2-i)^n \cdot (n!)^4} \right|} = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{|\sqrt{5}|^n} \left| \frac{(4n)!}{(n!)^4} \right|}$$

$$(4n)! = (4n)(4n-1)(4n-2) \ldots 4 \cdot 3 \cdot 2 \cdot 1$$

Wenn wir die Multiplikation in 4er Gruppen aufteilen:

$$\left.\begin{array}{l} [(4n)(4n-1)(4n-2)(4n-3)] \leq (4n)^4 \\ \vdots \\ [4 \cdot 3 \cdot 2 \cdot 1] \leq (4n)^4 \end{array}\right\} n\text{-Mal}$$

$$\Rightarrow (4n)! \leq (4n)^4 \cdot (4n)^4 \cdot \ldots = (4n)^{4n}$$

$$(n!)^4 = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

Wenn wir die Multiplikation in 4er Gruppen aufteilen

$$\left.\begin{array}{l} [n \cdot (n-1)(n-2)(n-3)] \leq n^4 \\ \vdots \\ [4 \cdot 3 \cdot 2 \cdot 1] \leq n^4 \end{array}\right\} n\text{-Mal}$$

$$\Rightarrow (n!)^4 \leq n^{4n}$$

Frage 2

$$\limsup_{n \to \infty} \sqrt[n]{\frac{1}{\sqrt{5}} \left| \frac{(4n)!}{(n!)^4} \right|} \leq \sqrt[n]{\frac{1}{\sqrt{5}} \left| \frac{(4n)^{4n}}{(n)^{4n}} \right|} = \sqrt[n]{\frac{1}{|\sqrt{5}|} \left| \frac{4^{4n} \cancel{n^{4n}}}{\cancel{n^{4n}}} \right|} = \frac{256}{\sqrt{5}}$$

$$\underline{\underline{R = \frac{1}{\frac{256}{\sqrt{5}}} = \frac{\sqrt{5}}{256}}}$$

Frage 3

$$z = (-\sqrt{3} + i)^7 \quad \| \; z = r^n \cdot e^{in\phi}$$

$$r = \sqrt{\sqrt{3}^2 + 1^2} = 2$$

$$\phi = \arctan\left(\frac{1}{-\sqrt{3}}\right) = -\frac{\pi}{6}$$

$$\Rightarrow z = 2^7 \cdot e^{-i\frac{7\pi}{6}} = 128\, e^{-i\frac{7\pi}{6}}$$

Serie 4

Frage 1

$$z^5 = -\frac{3}{2} + \frac{3^{\frac{3}{2}}}{2} i$$

$$r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}^3}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{27}{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3$$

$$\phi = \arctan\left(\frac{3^{\frac{3}{2}}/2}{-3/2}\right) + \pi = \frac{2\pi}{3} + 2\pi k$$

$$5\varphi = \frac{2\pi}{3} + 2\pi k$$

$$\varphi = \frac{2\pi}{15} + \frac{2\pi k}{5} \quad \Big\| \text{ mit } 0 \leq k \leq 4$$

$$z_1 = 3^{1/5} \cdot e^{i \cdot \left(\frac{2\pi}{15}\right)}$$

$$z_2 = 3^{1/5} \cdot e^{i \cdot \left(\frac{2\pi}{15} + \frac{2\pi}{5}\right)}$$

$$z_3 = 3^{1/5} \cdot e^{i \left(\frac{2\pi}{15} + \frac{4\pi}{5}\right)}$$

$$z_4 = 3^{1/5} \cdot e^{i \left(\frac{2\pi}{15} + \frac{6\pi}{5}\right)}$$

$$z_5 = 3^{1/5} \cdot e^{i \left(\frac{2\pi}{15} + \frac{8\pi}{5}\right)}$$

Frage 2

$$\mathrm{Log}\left(2^{3/2} + 2^{3/2} i\right) \quad \Big\| \quad \mathrm{Log}(z) = \log(|z|) + \mathrm{Arg}(z) \cdot i$$

$$|z| = \sqrt{\left(2^{3/2}\right)^2 + \left(2^{3/2}\right)^2} = \sqrt{2^3 + 2^3} = \sqrt{16} = 4$$

$$\mathrm{Arg}(z) = \frac{\pi}{4}$$

$$\Rightarrow \log(4) + \frac{\pi}{4} i$$

Serie 5

Frage 1

$u(x,y) = 4x^3y - 4xy^3$

$\frac{\partial u}{\partial x} = 12x^2y - 4y^3 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 24xy$

$\frac{\partial u}{\partial y} = 4x^3 - 12xy^2 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -24xy$

$= 0$

$v(x,y) = y^4 - 6x^2y^2 + x^4$

$\Rightarrow$ Nicht holomorph

$\frac{\partial v}{\partial x} = -12xy^2 + 4x^3 \Rightarrow \frac{\partial^2 v}{\partial x^2} = 12x^2$

$\frac{\partial v}{\partial y} = 4y^3 - 12x^2y = \frac{\partial^2 v}{\partial y^2} = 12y^2$

$\neq 0$

Frage 2

$f(z) = -z^4 \quad \| \; z = x + iy$

$= -(x+iy)^4 = -(x^4 + 4x^3iy - 6x^2y^2 - 4xiy^3 + y^4)$

$= -x^4 - 4x^3iy + 6x^2y^2 + 4xiy^3 - y^4$

$u(x,y) = -x^4 + 6x^2y^2 - y^4 \; ; \; v(x,y) = -4x^3iy + 4xiy^3$

$\frac{\partial u}{\partial x} = -4x^3 + 12xy^2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = -12x^2 + 12y^2$

0

$\frac{\partial u}{\partial y} = 12x^2y - 4y^3 \Rightarrow \frac{\partial^2 u}{\partial y^2} = 12x^2 - 12y^2$

$\Rightarrow$ holomorph

$\frac{\partial v}{\partial x} = -12x^2iy + 4iy^3 \Rightarrow \frac{\partial^2 v}{\partial x^2} = -24xiy$

0

$\frac{\partial v}{\partial y} = -4x^3i + 12xiy^2 \Rightarrow \frac{\partial^2 v}{\partial y^2} = 24xiy$

Frage 3

$\lim_{z \to 0} \left(\frac{z}{\bar{z}}\right)^2 \quad \| \; z = x + iy$

$\lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{(x+iy)}{(x-iy)}\right)^2 = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 + \cancel{2xiy} - y^2}{x^2 - \cancel{2xiy} - y^2}$ || Limes von x und y separat betrachten

$\lim_{\substack{x \to 0 \\ y = 0}} \frac{x^2}{x^2} = 1^2 = 1$

$\lim_{\substack{y \to 0 \\ x = 0}} \frac{-y^2}{-y^2} = 1^2 = 1$

Kontrolle für $x=y$ => da es für alle Richtungen gelten muss.

$z = x+ix$

$$\lim_{x\to\infty}\left(\frac{x+ix}{x-ix}\right)^2 = \lim_{x\to\infty}\frac{\cancel{x^2}+2ix^2-\cancel{x^2}}{\cancel{x^2}-2ix^2-\cancel{x^2}} = -1$$

=> Die Limes sind nicht gleich. Somit existiert der Limes nicht

Frage 5

$$f(z) = \frac{z-4}{z-3} = \frac{x+iy-4}{x+iy-3} = \frac{(x-4)+iy}{(x-3)+iy} \;\Big\|\; \frac{(x-3)-iy}{(x-3)-iy}$$

$$= \frac{(x-4)(x-3)-iy(x-4)+iy(x-3)+y^2}{(x-3)^2+y^2}$$

$$\Rightarrow \mathrm{Re}(f) = \frac{(x-4)(x-3)+y^2}{(x-3)^2+y^2} \;;\; \mathrm{Im}(f)\; \frac{-y(x-4)+y(x-3)}{(x-3)^2+y^2}$$

Serie 6

Frage 1

$z = t^2$

$\frac{dz}{dt} = 2t$

$dz = 2t\,dt$

$\int_0^2 (t^2)^2 \cdot 2t\,dt = \int_0^2 t^4\, 2t\,dt = 2\int_0^2 t^5\,dt = 2\left[\frac{t^6}{6}\right]_0^2 = \frac{2\cdot 64}{6} = \frac{128}{6} = \frac{64}{3}$

Frage 2

$z_1(t) = 2i + 3 + t(2i + 1 - 2i - 3) = 2i - 2t + 3$

$dz = -2\,dt$

$z_2(t) = 2i + 1 + t(1 - 2i - 1) = 2i - 2it + 1$

$dz = -2i\,dt$

$z_3(t) = 1 + t(3 - 1) = 1 + 2t$

$dz = 2\,dt$

$z_4(t) = 3 + t(2i + 3 - 3) = 3 + 2it$

$dz = 2i\,dt$

$|z_1|^2 = (3-2t)^2 + 2^2 = 13 - 12t + 4t^2$

$|z_2|^2 = (2-2t)^2 + 1^2 = 5 - 8t + 4t^2$

$|z_3|^2 = (1+2t)^2 = 1 + 2t + 4t^2$

$|z_4|^2 = 2t^2 + 3^2 = 9 + 4t^2$

$\int_0^1 (13 - 12t + 4t^2)(-2)\,dt = -2\int_0^1 (13 - 12t + 4t^2)\,dt = -2\left[\frac{4t^3}{3} - 6t^2 + 13t\right]_0^1$

$= -2\left[\frac{4}{3} - 6 + 13\right] = -\frac{50}{3}$

$\int_0^1 (5 - 8t + 4t^2)(-2i)\,dt = -2i\int_0^1 (5 - 8t + 4t^2)\,dt = -2i\left[\frac{4t^3}{3} - 4t^2 + 5t\right]_0^1$

$= -2i\left[\frac{4}{3} - 4 + 5\right] = -\frac{14}{3}i$

$$\int_0^1 (1+2t+4t^2)(2)\,dt = 2\cdot\int_0^1 1+2t+4t^2\,dt = 2\cdot\left[\frac{4t^3}{3}+t^2+t\right]_0^1$$

$$= 2\cdot\left[\frac{4}{3}+1+1\right] = \frac{26}{3}$$

$$\int_0^1 (9+4t^2)(2i)\,dt = 2i\int_0^1 9+4t^2\,dt = 2i\left[\frac{4t^3}{3}+9t\right]_0^1$$

$$2i\left[\frac{4}{3}+9\right] = \frac{62}{3}i$$

$$\Rightarrow \int_\Gamma |z|^2\,dz = -\frac{24}{3}+\frac{48}{3}i = -8+16i$$

Frage 3

$$z(t) = 3+6i+t(5+4i-3-6i) = 3+6i+2t-2it$$

$$dz = 2-2i\,dt$$

$$\int_0^1 (3-6i+2t+2it)(2-2i)\,dt = \int_0^1 6-12i+4t+4it-6i-12-4ti+4t\,dt$$

$$= \int_0^1 -6+8t-18i\,dt = \left[-6t+4t^2-18it\right]_0^1 = -2-18i$$

Frage 4

$$z(t) = \frac{\pi}{2}e^{it}$$

$$dz = \frac{\pi}{2}ie^{it}\,dt$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi}{2}e^{it}\right)^3\left(\frac{\pi}{2}ie^{it}\right)dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi^3}{8}e^{3it}\;\frac{\pi}{2}ie^{it}\,dt = \frac{\pi^4}{16}i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{4it}\,dt$$

$$= \frac{\pi^4}{16}i\left[\frac{e^{4it}}{4i}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1-1 = 0$$

Frage 5

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{|z-z_0|} = \lim_{z\to z_0}\frac{(z+3)^2-(z_0+3)^2}{|z-z_0|} = \lim_{z\to z_0}\frac{z^2+6z+9-z_0^3-6z_0-9}{z-z_0}$$

$$= \lim_{z\to z_0}\frac{(z-z_0)^2+6(z-z_0)}{z-z_0} = \lim_{z\to z_0}\frac{(z+z_0)\cancel{(z-z_0)}+6\cancel{(z-z_0)}}{\cancel{z-z_0}} = 2z_0+6$$

$$f(x,y) = ((x+iy)+3)^2 = (x+3)^2+2iy(x+3)-y^2 = x^2+6x+9+2ixy+6iy-y^2$$

$$= (x^2-y^2+6x+9)+i(2xy+6y)$$

$$\frac{\partial f_1}{\partial x} = 2x + 6 + 2yi$$

$$\frac{\partial f_1}{\partial y} = -2y + 2xi + 6i$$

$$u(x,y) = x^2 - y^2 + 6x + 9$$

$$v(x,y) = 2xy + 6y$$

$$\frac{\partial u}{\partial x} = 2x + 6$$

$$\frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial v}{\partial y} = 2x + 6$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \checkmark$$

$\Rightarrow$ Es gilt die Cauchy-Riemann Gleichung

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \checkmark$$

Seie 7

Frage 1

a) $z(t) = -2+2i+2\cdot e^{it}$

$2i = -2+2i+2e^{it}$ $\qquad$ $-4+2i = -2+2i+2e^{it}$

$e^{it} = 1 \to \arg(1)=0$ $\qquad$ $e^{it} = -1 \to \arg(-1)=\pi$

$dz = z'(t)\,dt = 2ie^{it}dt$

$\int_\gamma f(z)\,dz = \int_0^\pi (4(-2-2i+2e^{-it})+3)(2ie^{it})\,dt$

$= \int_0^\pi (-5-8i+8e^{-it})(2ie^{it})\,dt$

$= \int_0^\pi -10ie^{it}+16e^{it}+16i\;dt$

$= -10e^{it}-16ie^{it}+16it\Big|_0^\pi$

$= -10e^{i\pi}-16ie^{i\pi}+16i\pi+10+16i$

$= 10+16i+16i\pi+10+16i$

$= 20+32i+16i\pi$ $\qquad$ i

$\int_\eta f(z)\,dz = \int_0^{-\pi} (4(-2-2i+2e^{-it})+3)(2ie^{-it})\,dt$

$= -10e^{it}-16ie^{it}+16it\Big|_{-\pi}^{0}$

$= -10e^{-i\pi}-16ie^{-i\pi}-16i\pi+10+16i$

$= 10+16i+16i\pi+10+16i$

$= 20+32i+16i\pi$

Frage 2

Frage 3

$z(t) = 0+2e^{it}$

$2 = 2e^{it}$ $\qquad$ $2i = 2e^{it}$

$e^{it} = 1 \to \arg(1)=0$ $\qquad$ $e^{it} = i \to \arg(i)=\frac{\pi}{2}$

$$dz = z'(t)\,dt = 2ie^{it}\,dt$$

Alle drei Integrale haben die gleiche Lösung, da sie den gleichen Weg zurücklegen

$$\int_0^{\pi/2} \left(6(2e^{it})^2 + 4e^{it}\right)(2ie^{it})\,dt$$

$$= \int_0^{\pi/2} \left(24e^{2it} + 4e^{it}\right)(2ie^{it})\,dt$$

$$= \int_0^{\pi/2} 48ie^{3it} + 8ie^{2it}\,dt = 48e^{3it} + 8e^{2it}\Big|_0^{\pi/2} = -48i - 8 - 48 - 8 = -64 - 48i$$

Serie 8

Frage 1

$z_0^2 + 4 = 0$

$z_0 = \pm 2i$ || Laut $\gamma(t)$ ist $-2i$ ausserhalb vom Gebiet.

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z+4}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{e^{6i}}{-2i+4} = \frac{1}{\cancel{2\pi i}}\cdot \cancel{2\pi i}\cdot f(z_0) = \frac{e^{-3z_0}}{z_0+2i} = \underline{\underline{\frac{e^{-6i}}{4i}}}$$

Frage 2

$z_0 = 0$

$$\int_c \frac{\cos(z)}{z^2} = \frac{2\pi i}{1!} f^{(1)}(z_0) = 2\pi i \sin(0) = \underline{\underline{0}}$$

Frage 3

$z_0 = 0$

$$\int_c \frac{e^{3z}}{z}\,dz = \frac{2\pi i}{(n-1)!}\cdot 3^{(n-1)} e^{3\cdot 0} = \frac{2\pi i}{(n-1)!}\cdot 3^{(n-1)}$$

Für $n=0$ und $n<0$ ist das Integral 0, da die Funktion holomorph wird.

Frage 4

$$\int_c \frac{\sin(e^{-z})}{(z+1)(z^2-9)}\,dz = \int_c \frac{\sin(e^{-z})/(z^2-9)}{(z-1)}$$

$z_0 = -1$

$$\int_c \frac{\sin(e^{-z})}{(z+1)(z^2-9)}\,dz = \int_c \frac{\sin(e^{-z})/(z^2-9)}{(z+1)^1}\,dz = \frac{2\pi i}{0!}\cdot\frac{\sin(e)}{-8}$$