DISCLAIMER

Ich übernehme keine Haftung über mögliche Fehler in den Notizen. Es hat sicherlich ein paar drinnen.

Fehler können per Mail an jirruh@ethz.ch gemeldet werden.

Serie 1

Aufgabe 1

a) i) $6+6i$ ii) $-10+i$ iii) $3-39i$

iv) $\dfrac{(12+3i)}{(12+6i)} = \dfrac{(12+3i)(12-6i)}{144+36} = \dfrac{162-36i}{180}$

b) i) $\left.\begin{array}{l} -i \Rightarrow 1(\cos(-\frac{\pi}{2})+i\sin(-\frac{\pi}{2})) \\ 1-i \Rightarrow \sqrt{2}(\cos(-\frac{\pi}{4})+i\sin(-\frac{\pi}{4})) \end{array}\right\}$ $|z| = 1/\sqrt{2} = \frac{\sqrt{2}}{2}$, $\arg(z) = \left(-\frac{\pi}{2}+\frac{\pi}{4}\right) = -\frac{\pi}{4}$

$\Rightarrow z = \frac{\sqrt{2}}{2}(\cos(-\pi/4)+i\sin(-\pi/4))$

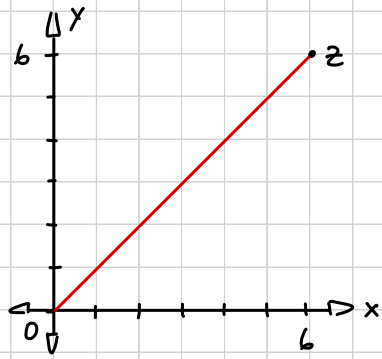
ii) $\left.\begin{array}{l} -2-2i \Rightarrow \sqrt{8}(\cos(-3\pi/4)+i\sin(-\pi/4)) \\ 1+\sqrt{3}i \Rightarrow 2(\cos(\pi/3)+i\sin(\pi/3)) \end{array}\right\}$ $|z| = \sqrt{8}/2$, $\arg(z) = (-3\pi/4-\pi/3) = -13\pi/12 = 11\pi/12$

$\Rightarrow z = \frac{\sqrt{8}}{2}(\cos(11\pi/12)+i\sin(11\pi/12))$

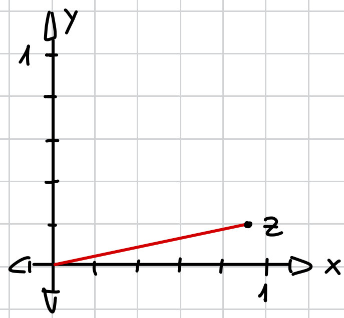
iii) $\left.\begin{array}{l} 1+\sqrt{3}i \Rightarrow 2(\cos(\pi/3)+i\sin(\pi/3)) \\ \sqrt{3}+i \Rightarrow 2(\cos(\pi/6)+i\sin(\pi/6)) \end{array}\right\}$ $|z| = 1$, $\arg(z) = (\pi/3-\pi/6) = \pi/6$

$z = 1(\cos(\pi/6)+i\sin(\pi/6))$

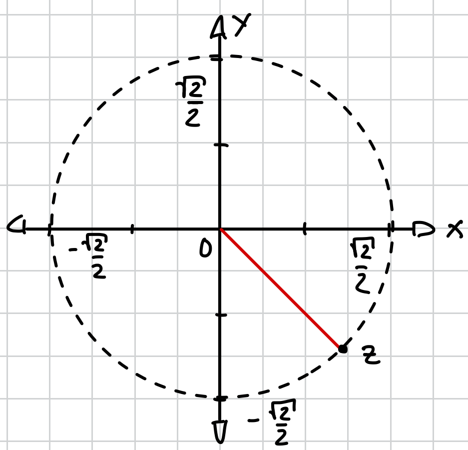
c) ai)



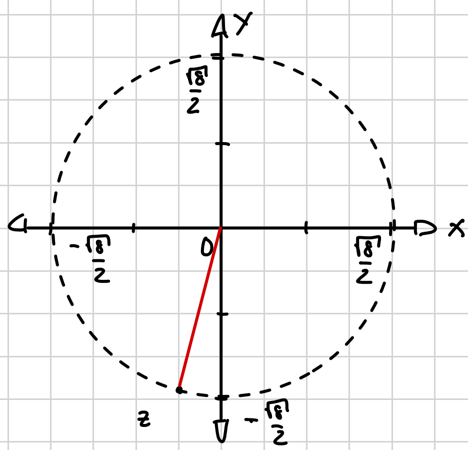
aiv)



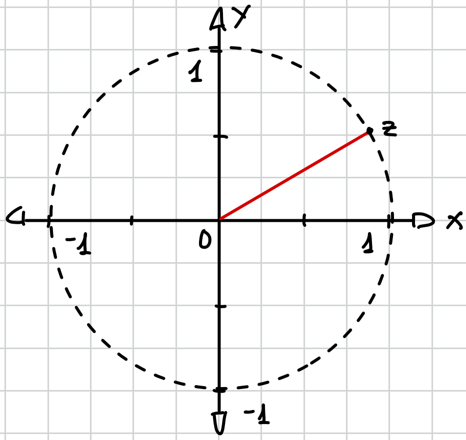
bi)



bii)

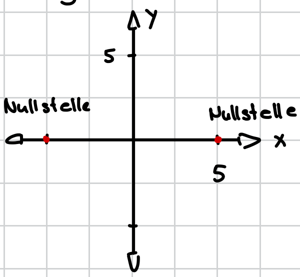


b iii)



Aufgabe 2

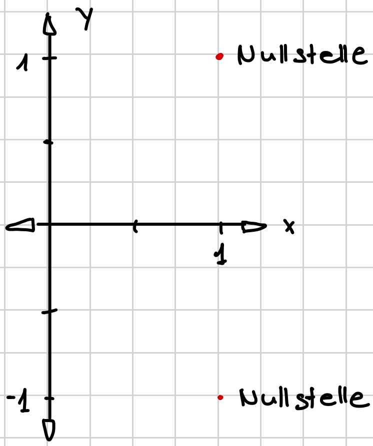
i) $z = \pm 5$



ii) $z^2 - 2z + 2 \stackrel{!}{=} 0$

$z_{1,2} = 1 \pm \sqrt{1 - 2}$

$= 1 \pm i$



iii) $z^3 + z^2 - 2 : z - 1 = z^2 + 2z + 2$

$z^3 - z^2$

$/ \quad 2z^2$

$2z^2 - 2z$

$/ \quad 2z - 2$

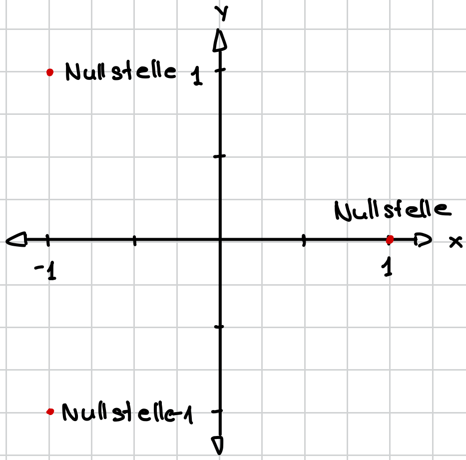
$2z - 2$

$z_1 = 1$

$z^2 + 2z + 2 \stackrel{!}{=} 0$

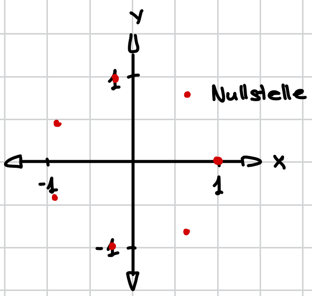
$z_{2,3} = -1 \pm \sqrt{1 - 2}$

$= -1 \pm i$



iv) $z = 1(\cos(7\varphi) + i \cdot \sin(7\varphi))$

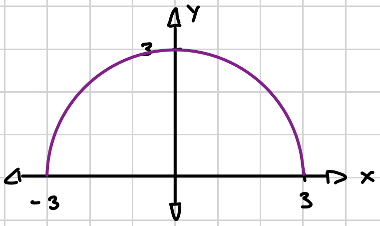
$7\varphi = 2\pi n \Rightarrow \varphi = \frac{2\pi n}{7}$



Serie 2

Aufgabe 1

i) $M = \{ z \in \mathbb{C} \mid |z| = 3,\ \mathrm{Im}(z) \geq 0 \}$



ii) $M = \left\{ z \in \mathbb{C} \;\middle|\; \frac{|z+2-2i|}{|z+i|} = 2 \right\}$

$= \left\{ z \in \mathbb{C} \;\middle|\; \frac{|x+2+i(y-2)|}{|x+i(y+1)|} = 2 \right\}$

$(|x+2+i(y-2)|)^2 = (2|x+i(y+1)|)^2$

$(x+2)^2 + (y-2)^2 = 4(x^2+(y+1)^2)$

$x^2+4x+\cancel{4}+y^2-4y+4 = 4x^2+4y^2+8y+\cancel{4}$

$-3x^2+4x+4 = 3y^2+12y$

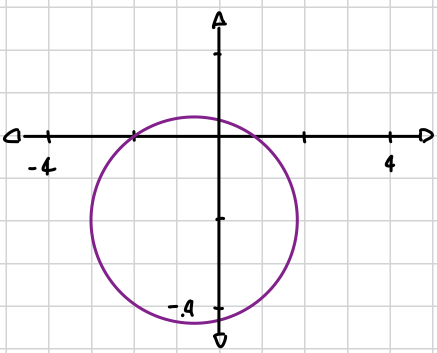
$3x^2-4x+3y^2+12y = 4$

$x^2 - \frac{4}{3}x + y^2 + 4y = \frac{4}{3}$

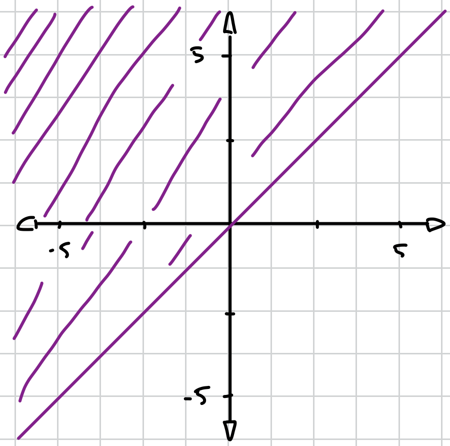
$(x-\frac{2}{3})^2 + (y+2)^2 = \frac{52}{9}$

$M = \{ z \in \mathbb{C} \mid |z - \frac{2}{3} + 2i| = \frac{2\sqrt{13}}{3}$

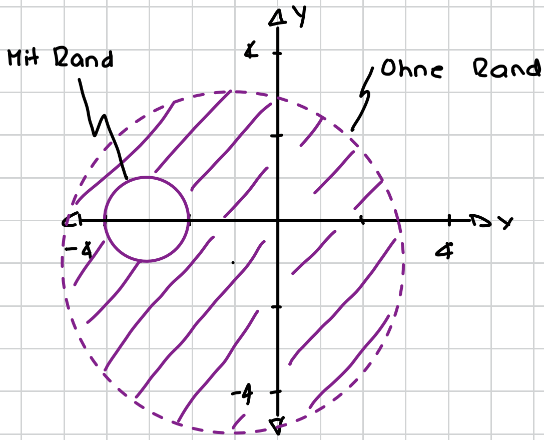
Nur der Rand



iii) $M = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) \geq \mathrm{Re}(z) \}$



iv) $M = \{ z \in \mathbb{C} \mid |z-3| \geq 1 \text{ und } |z-1-i| < 4 \}$



Serie 3

Aufgabe 1

a) i) $\lim_{|n|\to\infty} \cos(in) = \lim_{|n|\to\infty} \cosh(n) = \infty$

ii) $\lim_{n\to\infty} 1+(-1)^n \cdot \frac{i}{n} = 1 + \lim_{n\to\infty} (-1^n \cdot \frac{i}{n}) = 1+0i = 1$ || Wenn $n \to \infty$ wird $\frac{i}{n} \to 0$

iii) $\lim_{n\to\infty} \frac{(n+2\pi i)^n}{n^n} = \lim_{n\to\infty} \frac{n^n}{n^n} = 1$ || $2\pi i$ hat kein Einfluss aufs Ergebnis wenn $n \to \infty$

iv) $\lim_{n\to\infty} \operatorname{Arg}(1+(-1)^n \frac{i}{n})$ || $\lim_{n\to\infty} 1+(-1)^n \frac{i}{n} = 1$

$\operatorname{Arg}(1) = 0$

b)

$$\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(\pi i)^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{1}{2^n} + \sum_{n=0}^{\infty} \frac{(\pi i)^n}{n!} = 2 + \exp(\pi i) = 2 + e^{\pi i}$$

Aufgabe 2

$\exp(it) = \cos(t) + i \cdot \sin(t)$

$\exp(z_1 + z_2) = \exp(z_1) \cdot \exp(z_2)$ || $z_1 = ix$; $z_2 = iy$

$\exp(i(x+y)) = \exp(ix) \cdot \exp(iy)$ || Eulerformel

$\exp(i(x+y)) = (\cos(x) + i\sin(x)) \cdot (\cos(y) + i\sin(y))$

$\cos(x+y) + i\sin(x+y) = (\cos(x) + i\sin(x)) \cdot (\cos(y) + i\sin(y))$

$\cos(x+y) + i\sin(x+y) = \cos(x)\cos(y) + \cos(x) i \sin(y) + i\sin(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x+y) + i\sin(x+y) = (\cos(x)\cos(y) - \sin(x)\sin(y)) + i(\cos(x)\sin(y) + \cos(y)\sin(x))$

$\Rightarrow \cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\Rightarrow \sin(x+y) = \cos(x)\sin(y) + \cos(y)\sin(x)$

|| Ich bin zu faul um es für x-iy zu beweisen :-)

$\square$

Aufgabe 3

i) $\lim_{z\to 0} \frac{\bar{z} + z^2}{z}$ || $z = x+iy$

$\lim_{z\to 0} \frac{x - iy + x^2 + 2xiy - y}{x+iy}$ || Wir müssen den Grenzwert von Re und Im seperat betrachten

$\lim_{y\to 0} \frac{x - iy + x^2 + 2xiy - y^2}{x+iy}$ || $x=0$

$\lim_{y\to 0} \frac{-iy - y^2}{iy} = \lim_{y\to 0} -1 - \frac{y^2}{iy} = \lim_{y\to 0} -1 - \frac{y}{i} = -1$

Aufgabe 3

$$\lim_{x\to 0} \frac{x-iy+x^2+2xiy-y^2}{x+iy} \quad \Big\| \; y=0$$

$$\lim_{x\to 0} \frac{x+x^2}{x} = \lim_{x\to 0} 1+x = 1$$

$z_0 = 1-i \neq 0$ ‖ Somit existiert der Limes nicht.

ii) $$\lim_{z\to 0} \frac{\cos(z)-1}{z^2} \quad \Big\| \; \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \quad \text{(Taylorreihe)}$$

$$\lim_{z\to 0} \frac{-\frac{z^2}{2!} + \frac{z^4}{4!} + \dots}{z^2} = \underline{\underline{-\frac{1}{2}}}$$

iii) $$\lim_{z\to 0} \frac{\sin(z)}{\overline{z}} \quad \Big\| \; z = x+iy$$

$$\lim_{z\to 0} \frac{\sin(x+iy)}{x-iy}$$ ‖ Wir müssen den Grenzwert von Re und Im seperat betrachten

$$\lim_{x\to 0} \frac{\sin(x+iy)}{x-iy} \quad \Big\| \; y=0$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \quad \Big\| \; \lim_{x\to 0} \frac{\sin(x)'}{x'} = \frac{\cos(x)}{1} = 1 \quad \text{(Bernoulli - l'Hôpital)}$$

$$\lim_{y\to 0} \frac{\sin(x+iy)}{x-iy} \quad \Big\| \; x=0$$

$$\lim_{y\to 0} \frac{\sin(iy)}{-iy} = \lim_{y\to 0} \frac{\cancel{i}\cdot\sin(y)}{-\cancel{i}y} = \lim_{y\to 0} -\frac{\sin(y)}{y} = -1 \quad \Big\| \; \text{Bernoulli - l'Hôpital}$$

$\Rightarrow z_0 = 1-i \neq 0$ ‖ Somit existiert der Limes nicht.

Serie 4

Aufgabe 1

a) i) $e^{i} = \cos(1) + i\sin(1)$

ii) $e^{1-2i} = e \cdot e^{-2i}$

$e^{-2i} = \cos(-2) + i\sin(-2)$

$\Rightarrow e^{1-2i} = e\cos(-2) + e \cdot i \sin(-2)$

iii) $\mathrm{Log}(1+i) = \log(\sqrt{2}) + i\frac{\pi}{4}$

b) i) $\cos(10i) = \cosh(10)$

$\cosh(10) = \frac{e^{x} + e^{-x}}{2}$

ii) $\sin(5+5i) = \sin(5)\cos(5i) + \cos(5)\sin(5i)$

$\cos(5i) = \cosh(5) = \frac{e^{5} + e^{-5}}{2}$

$\sin(5i) = \sinh(5) = \frac{e^{5} - e^{-5}}{2}$

$\sin(5+5i) = \sin(5)\frac{e^{5}+e^{-5}}{2} + \cos(5)\frac{e^{5}-e^{-5}}{2}$

iii) $\sin(2-i) = \sin(2)\cos(i) - \cos(2)\sin(i)$

$\cos(i) = \frac{e^{1}+e^{-1}}{2}$

$\sin(i) = \frac{e^{1}-e^{-1}}{2}$

$\sin(2-i) = \sin(2)\frac{e^{1}+e^{-1}}{2} - \cos(2)\frac{e^{1}-e^{-1}}{2}$

Aufgabe 2

$\mathrm{Log}(z_1 \cdot z_2) \neq \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$

|| Wir suchen z_1 und z_2, sodass die Summe der Arg() > π sind aber das Arg() von $z_1 \cdot z_2$ kleiner als π ist.

$$\Rightarrow \begin{cases} z_1 = -1 \\ z_2 = i \end{cases}$$

Aufgabe 3

$(\cos(\phi) + i\sin(\phi))^n = \cos(n\phi) + i \cdot \sin(n\phi)$ ‖ Beweis durch Induktion

$n = 0$

$(\cos(\phi) + i\sin(\phi))^n = \cos(n\phi) + i \cdot \sin(n\phi)$

$1 = 1 + 0$ ✓

$n = 1$

$(\cos(\phi) + i\sin(\phi))^1 = \cos(\phi) + i\sin(\phi)$ ✓

$n = k + 1$

$(\cos(\phi) + i\sin(\phi))^{k+1} = (\cos(\phi) + i\sin(\phi))^k \cdot (\cos(\phi) + i\sin(\phi))$

$= (\cos(k\phi) + i\sin(k\phi))(\cos(\phi) + i\sin(\phi))$

$= \cos(k\phi)\cos(\phi) + \cos(k\phi)i\sin(\phi) + i\sin(k\phi)\cos(\phi) - \sin(k\phi)\sin(\phi)$ ‖ Kosinus & Sinus Satz

$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$\sin(A+B) = \cos(A)\sin(B) + \cos(B)\sin(A)$

$\Rightarrow \cos(k\phi + k) + \sin(k\phi + k) = \cos((k+1)\phi) + \sin((k+1)\phi)$

$\square$

Serie 5

Aufgabe 1

$$f'(z) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} \quad \| \; \Delta z = z - z_0$$

i) $f(z) = 3z^3 + z - 3$

$$f'(z) = \lim_{\Delta z \to 0} \frac{3(z+\Delta z)^3 + (z+\Delta z) - 3 - (3z^3 + z - 3)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3(z^3 + 3z^2\Delta z + 3z\Delta z^2 + \Delta z^3) + (z+\Delta z) - 3 - (3z^3+z-3)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{3z^3} + 9z^2\Delta z + 9z\Delta z^2 + 3\Delta z^3 + \cancel{z} + \Delta z \cancel{-3} - \cancel{3z^3} - \cancel{z} \cancel{+3}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{9z^2\Delta z + 9z\Delta z^2 + 3\Delta z^3 + \Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{\Delta z}(9z^2 + 9z\Delta z + 3\Delta z^2 + 1)}{\cancel{\Delta z}}$$

$$= \lim_{\Delta z \to 0} 9z^2 + 9z\Delta z + 3\Delta z^2 + 1$$

$$\Rightarrow \underline{\underline{f'(z) = 9z^2 + 1}}$$

ii) $f(z) = \sin(\mathrm{Re}(z)) = \sin(x)$ $\|$ Grundsätzlich wäre diese Gleichung ableitbar aber es ist in einem Bereich ableitbar.

$f'(z) = \cos(x)$

$\tilde{x} \in [\pi/2 + k\pi]$

$\Rightarrow$ Wenn $x \in \tilde{x}$ dann existiert die Ableitung, sonst nicht

iii) $f(z) = \frac{1}{z^2}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\frac{1}{(z+\Delta z)^2} - \frac{1}{z^2}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\frac{z^2 - (z+\Delta z)^2}{z^2(z+\Delta z)^2}}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 - (z+\Delta z)^2}{\Delta z(z^2(z+\Delta z)^2)}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z^2} - \cancel{z^2} - 2z\Delta z - \Delta z^2}{\Delta z(z^2(z+\Delta z)^2)} = \lim_{\Delta z \to 0} \frac{\cancel{\Delta z}(-2z - \Delta z)}{\cancel{\Delta z}(z^2(z+\Delta z)^2)}$$

$$\Rightarrow \underline{\underline{f'(z) = \frac{-2z}{z^4} = -\frac{2}{z^3}}}$$

iv) $f(z) = e^{-\pi z^2}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{e^{-\pi(z+\Delta z)^2} - e^{-\pi z^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{e^{-\pi(z^2+2z\Delta z+\Delta z^2)} - e^{-\pi z^2}}{\Delta z} = \lim_{\Delta z \to 0} \frac{e^{-\pi z^2} \cdot e^{-\pi(2z\Delta z+\Delta z^2)} - e^{-\pi z^2}}{\Delta z}$$

$\Rightarrow$ Wir verwenden die Aproximation, dass $e^{-\pi(2z\Delta z+\Delta z^2)} \approx 1 - \pi(2z\Delta z+\Delta z^2)$ für $\Delta z \to 0$ (Taylor Series $\to e^x = 1 + x + \frac{x^2}{2!} + \ldots$)

$$= \lim_{\Delta z \to 0} \frac{e^{-\pi z^2}(\cancel{1} - \pi(2z\Delta z+\Delta z^2) - \cancel{1})}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{e^{-\pi z^2}(-\pi(2z\cancel{\Delta z}+\Delta z^2))}{\cancel{\Delta z}}$$

$$\Rightarrow \underline{\underline{f'(z) = -2\pi z \cdot e^{-\pi z^2}}}$$

Aufgabe 2

$g(t) = f(\gamma(t))$

$$g'(t) = \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(\gamma(t+\Delta t)) - f(\gamma(t))}{\Delta t} \quad \Big\| \cdot \frac{\gamma(t+\Delta t) - \gamma(t)}{\gamma(t+\Delta t) - \gamma(t)}$$

$$= \lim_{\gamma(t+\Delta t) \to \gamma(t)} \frac{f(\gamma(t+\Delta t)) - f(\gamma(t))}{\gamma(t+\Delta t) - \gamma(t)} \cdot \lim_{\Delta t \to 0} \frac{\gamma(t+\Delta t) - \gamma(t)}{\Delta t} \quad \Big\| \Delta\gamma = \gamma(t+\Delta t) - \gamma(t) \Rightarrow \Delta z = z - z_0$$

$$\Rightarrow \underline{\underline{g'(t) = f'(\gamma(t)) \cdot \dot{\gamma}(t)}}$$

Aufgabe 3

Cauchy-Riemann Gleichung: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

a) $u(x,y) := \sin(x^2-y^2)\cosh(2xy)$

$$\frac{\partial u}{\partial x} = 2x \cdot \cos(x^2-y^2)\cosh(2xy) + 2y \cdot \sin(x^2-y^2)\sinh(2xy)$$

$$\frac{\partial u}{\partial y} = -2y \cdot \cos(x^2-y^2)\cosh(2xy) + 2x \cdot \cos(x^2-y^2)\sinh(2xy)$$

$v(x,y) = -\cos(x^2-y^2)\sinh(2xy)$

$$\frac{\partial v}{\partial x} = 2x \cdot \sin(x^2-y^2)\sinh(2xy) - 2y\cos(x^2-y^2)\cosh(2xy)$$

$$\frac{\partial v}{\partial y} = -2y \cdot \sin(x^2-y^2)\sinh(2xy) - 2x \cdot \cos(x^2-y^2)\cosh(2xy)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \checkmark$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \checkmark$$

$\Rightarrow$ Da die Cauchy-Riemann Gleichungen gelten, ist die Gleichung holomorph

b) $u(x,y) := e^{x^2-y^2} \cos(2xy)$

$$\frac{\partial u}{\partial x} = 2x \cdot e^{x^2-y^2} \cdot \cos(2xy) - 2y \cdot e^{x^2-y^2} \cdot \sin(2xy)$$

$$\frac{\partial u}{\partial y} = -2y \cdot e^{x^2-y^2} \cdot \cos(2xy) - 2x \cdot e^{x^2-y^2} \cdot \sin(2xy)$$

$v(x,y) := e^{x^2-y^2} \sin(2xy)$

$$\frac{\partial v}{\partial x} = 2x\, e^{x^2-y^2} \cdot \sin(2xy) + 2y \cdot e^{x^2-y^2} \cdot \cos(2xy)$$

$$\frac{\partial v}{\partial y} = -2y \cdot e^{x^2-y^2} \cdot \sin(2xy) + 2x \cdot e^{x^2-y^2} \cdot \cos(2xy)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \checkmark$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \checkmark$$

$\Rightarrow$ Da die Cauchy-Riemann Gleichung gelten, Ist die Gleichung holomorph

Aufgabe 4

$x = r \cdot \sin(\varphi)$

$y = r \cdot \cos(\varphi)$

$r = \sqrt{x^2+y^2}$

$\varphi = \arctan\left(\frac{x}{y}\right)$

$\tilde{u}(x,y) = \tilde{u}(r \cdot \sin(\varphi), r \cdot \cos(\varphi))$

$\tilde{v}(x,y) = \tilde{v}(r \cdot \sin(\varphi), r \cdot \cos(\varphi))$

$$\frac{\partial \tilde{u}}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial r}$$ || Kettenregel

$$\frac{\partial \tilde{u}}{\partial \varphi} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial \varphi}$$

$$\frac{\partial \tilde{v}}{\partial r} = \frac{\partial v}{\partial x} \frac{\partial v}{\partial r} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial r}$$

$$\frac{\partial \tilde{v}}{\partial \varphi} = \frac{\partial v}{\partial x} \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial \varphi}$$

$$\frac{\partial \tilde{u}}{\partial r} = \frac{\partial u}{\partial x} \cdot \cos(\varphi) + \frac{\partial u}{\partial y} \cdot \sin(\varphi) \Rightarrow \frac{\partial v}{\partial y} \cdot \cos(\varphi) - \frac{\partial v}{\partial x} \cdot \sin(\varphi)$$

$$\frac{\partial \tilde{u}}{\partial \varphi} = \frac{\partial u}{\partial x} \cdot -r \cdot \sin(\varphi) + \frac{\partial u}{\partial y} \cdot r \cdot \cos(\varphi) \Rightarrow \frac{\partial v}{\partial y} \cdot -r \cdot \sin(\varphi) - \frac{\partial v}{\partial x} \cdot r \cdot \cos(\varphi)$$

$$\frac{\partial \tilde{v}}{\partial r} = \frac{\partial v}{\partial x} \cdot \cos(\varphi) + \frac{\partial v}{\partial y} \cdot \sin(\varphi) \Rightarrow -\frac{\partial u}{\partial y} \cdot \cos(\varphi) + \frac{\partial u}{\partial x} \cdot \sin(\varphi)$$

$$\frac{\partial \tilde{v}}{\partial \varphi} = \frac{\partial v}{\partial x} \cdot -r \cdot \sin(\varphi) + \frac{\partial v}{\partial y} \cdot r \cdot \cos(\varphi) \Rightarrow -\frac{\partial u}{\partial y} \cdot -r \cdot \sin(\varphi) + \frac{\partial u}{\partial x} \cdot r \cdot \cos(\varphi)$$

$$r \cdot \frac{\partial \hat{u}}{\partial r} = r \cdot \left(\frac{\partial v}{\partial y} \cdot \cos(\varphi) - \frac{\partial v}{\partial x} \cdot \sin(\varphi) \right) = \frac{\partial v}{\partial y} \cdot r \cdot \cos(\varphi) - \frac{\partial v}{\partial x} \cdot r \cdot \sin(\varphi) = \frac{\partial \tilde{v}}{\partial r} \ \checkmark$$

$$-r \cdot \frac{\partial \tilde{v}}{\partial r} = -r \left(-\frac{\partial u}{\partial y} \cdot \cos(\varphi) + \frac{\partial u}{\partial x} \cdot \sin(\varphi) \right) = \frac{\partial u}{\partial y} \cdot r \cdot \cos(\varphi) - \frac{\partial u}{\partial x} \cdot \sin(\varphi) = \frac{\partial \tilde{u}}{\partial \varphi} \ \checkmark$$

$$\square$$

DISCLAIMER

Die Notizen zu den Stack Aufgaben sind auf meine Werte angepasst. Die Werte können von deinen Aufgaben abweichen.

Serie 1

Frage 1

$(8-i)(5i+3) = 40i+24+5-3i = 37i+29$

$i\,(5i+3) = -5+3i$

$\bar{w} = -5i+3$

$\frac{8-i}{5i+3} = \frac{(8-i)(5i-3)}{-34} = \frac{40i-24+5+3i}{-34} = -\frac{43i}{34} + \frac{19}{34}$

$|z| \sqrt{8^2+1^2} = \sqrt{65}$

Frage 2

$\cos(-\frac{\pi}{4})>0\,;\ \cos(\frac{\pi}{4})>0 \Rightarrow Re(z)>0$

$\sin(-\frac{\pi}{4})<0\,;\ \sin(\frac{\pi}{4})>0 \Rightarrow Im(z)>/=/<\ 0$

Frage 3

$(4+2i)(4+2i) = 16+8i+8i-4 = 12+16i$

$(12+16i)(4+2i) = 48+24i+64i-32 = 16+88i$

Bemerkung: Teilweise ist es einfacher potenzierte komplexe Gleichungen in Polarform zu berechnen.

Serie 2

Serie 3

Frage 1

$\mathrm{Im}(z) = 0$ || Der Im von z muss 0 sein damit $z \in \mathbb{R}$

$$z = e^{\frac{3\pi}{4}i} \cdot (\sqrt{2} + bi)$$

$$= \left(\cos\left(\tfrac{3\pi}{4}\right) + i \sin\left(\tfrac{3\pi}{4}\right)\right) \cdot (\sqrt{2} + bi)$$

$$= \sqrt{2}\cos\left(\tfrac{3\pi}{4}\right) + bi\cos\left(\tfrac{3\pi}{4}\right) + \sqrt{2}\,i\sin\left(\tfrac{3\pi}{4}\right) - b\sin\left(\tfrac{3\pi}{4}\right) \quad \| \ \mathrm{Im}(z) = 0$$

$$\Rightarrow bi\cos\left(\tfrac{3\pi}{4}\right) + \sqrt{2}\,i\sin\left(\tfrac{3\pi}{4}\right) = 0$$

$$bi\cos\left(\tfrac{3\pi}{4}\right) = -\sqrt{2}\,i\sin\left(\tfrac{3\pi}{4}\right)$$

$$\cancel{-}\,\cancel{bi}\,\frac{\sqrt{2}}{2} = \cancel{-i}$$

$$b = \frac{2\sqrt{2}}{2}$$

Frage 2

$$c_n = \frac{(4n)!}{(-2-i)^n \cdot (n!)^4}$$

$$R = \frac{1}{\limsup\limits_{n \to \infty} \sqrt[n]{|C_n|}}$$

$$\limsup_{n \to \infty} \sqrt[n]{\left| \frac{(4n)!}{(-2-i)^n \cdot (n!)^4} \right|} = \limsup_{n \to \infty} \sqrt[n]{\frac{1}{|\sqrt{5}|^n} \left| \frac{(4n)!}{(n!)^4} \right|}$$

$$(4n)! = (4n)(4n-1)(4n-2) \dots 4 \cdot 3 \cdot 2 \cdot 1$$

Wenn wir die Multiplikation in 4er Gruppen aufteilen:

$$\left.\begin{array}{l} [(4n)(4n-1)(4n-2)(4n-3)] \leq (4n)^4 \\ \vdots \\ [4 \cdot 3 \cdot 2 \cdot 1] \leq (4n)^4 \end{array}\right\} n\text{-Mal}$$

$$\Rightarrow (4n)! \leq (4n)^4 \cdot (4n)^4 \cdot \ldots = (4n)^{4n}$$

$$(n!)^4 = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

Wenn wir die Multiplikation in 4er Gruppen aufteilen

$$\left.\begin{array}{l} [n \cdot (n-1)(n-2)(n-3)] \leq n^4 \\ \vdots \\ [4 \cdot 3 \cdot 2 \cdot 1] \leq n^4 \end{array}\right\} n\text{-Mal}$$

$$\Rightarrow (n!)^4 \leq n^{4n}$$

Frage 2

$$\limsup_{n\to\infty} \sqrt[n]{\frac{1}{\sqrt{5}} \left| \frac{(4n)!}{(n!)^4} \right|} \leq \sqrt[n]{\frac{1}{\sqrt{5}} \left| \frac{(4n)^{4n}}{(n)^{4n}} \right|} = \sqrt[n]{\frac{1}{|\sqrt{5}|} \left| \frac{4^{4n} \cancel{n^{4n}}}{\cancel{n^{4n}}} \right|} = \frac{256}{\sqrt{5}}$$

$$R = \frac{1}{\frac{256}{\sqrt{5}}} = \frac{\sqrt{5}}{256}$$

Frage 3

$$z = (-\sqrt{3} + i)^7 \quad \| \; z = r^n \cdot e^{in\phi}$$

$$r = \sqrt{\sqrt{3}^2 + 1^2} = 2$$

$$\phi = \arctan\left(\frac{1}{-\sqrt{3}}\right) = -\frac{\pi}{6}$$

$$\Rightarrow z = 2^7 \cdot e^{-i\frac{7\pi}{6}} = 128\, e^{-i\frac{7\pi}{6}}$$

Serie 4

Frage 1

$$z^5 = -\frac{3}{2} + \frac{3^{\frac{3}{2}}}{2} i$$

$$r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}^3}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{27}{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3$$

$$\phi = \arctan\left(\frac{3^{\frac{3}{2}}/2}{-3/2}\right) + \pi = \frac{2\pi}{3} + 2\pi k$$

$$5\varphi = \frac{2\pi}{3} + 2\pi k$$

$$\varphi = \frac{2\pi}{15} + \frac{2\pi k}{5} \quad \| \text{ mit } 0 \leq k \leq 4$$

$$z_1 = 3^{1/5} \cdot e^{i \cdot \left(\frac{2\pi}{15}\right)}$$

$$z_2 = 3^{1/5} \cdot e^{i \cdot \left(\frac{2\pi}{15} + \frac{2\pi}{5}\right)}$$

$$z_3 = 3^{1/5} \cdot e^{i \left(\frac{2\pi}{15} + \frac{4\pi}{5}\right)}$$

$$z_4 = 3^{1/5} \cdot e^{i \left(\frac{2\pi}{15} + \frac{6\pi}{5}\right)}$$

$$z_5 = 3^{1/5} \cdot e^{i \left(\frac{2\pi}{15} + \frac{8\pi}{5}\right)}$$

Frage 2

$$\mathrm{Log}\left(2^{3/2} + 2^{3/2} i\right) \quad \| \; \mathrm{Log}(z) = \log(|z|) + \mathrm{Arg}(z) \cdot i$$

$$|z| = \sqrt{\left(2^{3/2}\right)^2 + \left(2^{3/2}\right)^2} = \sqrt{2^3 + 2^3} = \sqrt{16} = 4$$

$$\mathrm{Arg}(z) = \frac{\pi}{4}$$

$$\Rightarrow \log(4) + \frac{\pi}{4} i$$

Serie 5

Frage 1

$u(x,y) = 4x^3y - 4xy^3$

$\frac{\partial u}{\partial x} = 12x^2y - 4y^3 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 24xy$

$\frac{\partial u}{\partial y} = 4x^3 - 12xy^2 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -24xy$

$= 0$

$v(x,y) = y^4 - 6x^2y^2 + x^4$

$\Rightarrow$ Nicht holomorph

$\frac{\partial v}{\partial x} = -12xy^2 + 4x^3 \Rightarrow \frac{\partial^2 v}{\partial x^2} = 12x^2$

$\frac{\partial v}{\partial y} = 4y^3 - 12x^2y = \frac{\partial^2 v}{\partial y^2} = 12y^2$

$\neq 0$

Frage 2

$f(z) = -z^4 \quad \| \; z = x + iy$

$= -(x+iy)^4 = -(x^4 + 4x^3iy - 6x^2y^2 - 4xiy^3 + y^4)$

$= -x^4 - 4x^3iy + 6x^2y^2 + 4xiy^3 - y^4$

$u(x,y) = -x^4 + 6x^2y^2 - y^4 \; ; \; v(x,y) = -4x^3iy + 4xiy^3$

$\frac{\partial u}{\partial x} = -4x^3 + 12xy^2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = -12x^2 + 12y^2$

$\frac{\partial u}{\partial y} = 12x^2y - 4y^3 \Rightarrow \frac{\partial^2 u}{\partial y^2} = 12x^2 - 12y^2$

0

$\Rightarrow$ holomorph

$\frac{\partial v}{\partial x} = -12x^2iy + 4iy^3 \Rightarrow \frac{\partial^2 v}{\partial x^2} = -24xiy$

$\frac{\partial v}{\partial y} = -4x^3i + 12xiy^2 \Rightarrow \frac{\partial^2 v}{\partial y^2} = 24xiy$

0

Frage 3

$\lim_{z \to 0} \left(\frac{z}{\bar{z}}\right)^2 \quad \| \; z = x + iy$

$\lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{(x+iy)}{(x-iy)}\right)^2 = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 + 2\cancel{xiy} - \cancel{y^2}}{x^2 - 2\cancel{xiy} - \cancel{y^2}} \quad \|$ Limes von x und y seperat betrachten

$\lim_{\substack{x \to 0 \\ y = 0}} \frac{x^2}{x^2} = 1^2 = 1$

$\lim_{\substack{y \to 0 \\ x = 0}} \frac{-y^2}{-y^2} = 1^2 = 1$

Kontrolle für $x=y$ ⇒ da es für alle Richtungen gelten muss.

$z = x + ix$

$$\lim_{x\to\infty}\left(\frac{x+ix}{x-ix}\right)^2 = \lim_{x\to\infty}\frac{\cancel{x^2}+2ix^2-\cancel{x^2}}{\cancel{x^2}-2ix^2-\cancel{x^2}} = -1$$

⇒ Die Limes sind nicht gleich. Somit existiert der Limes nicht

Frage 5

$$f(z) = \frac{z-4}{z-3} = \frac{x+iy-4}{x+iy-3} = \frac{(x-4)+iy}{(x-3)+iy} \quad \Bigg\| \; \frac{(x-3)-iy}{(x-3)-iy}$$

$$= \frac{(x-4)(x-3) - iy(x-4) + iy(x-3) + y^2}{(x-3)^2 + y^2}$$

$$\Rightarrow \operatorname{Re}(f) = \frac{(x-4)(x-3)+y^2}{(x-3)^2+y^2} \; ; \; \operatorname{Im}(f) \; \frac{-y(x-4) + y(x-3)}{(x-3)^2+y^2}$$